Optimisation

Rubén Pérez Sanz

Universitat Autònoma de Barcelona

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INTRODUCTION

INTUITION

In economics, agents are assumed to be endowed with a **payoff function**, which is nothing else than an ordering of their preferences over the results of their actions.

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At the same time, agents are supposed to take **rational choices**, meaning that they maximised these payoff functions.

For example:

- Consumers are meant to maximise their utility over purchases
- Firms are supposed to maximise profits over investments
- Parties maximise votes over programmes
- and so on...

Let $f(\mathbf{x})$ be a function of many variables defined on a set X and let S be a subset of X. The point $\mathbf{x}^* \in S$ solves the problem

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$$\text{subject to } \mathbf{x} \in S$$

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$$f(\mathbf{x}) \leq f(\mathbf{x}^*) \quad \forall \mathbf{x} \in S$$

In this case we say that $\mathbf{x}^*$ is a **maximiser** of $f(\mathbf{x})$ subject to the constraint $\mathbf{x} \in S$, and that $f(\mathbf{x}^*)$ is the **maximum** (or maximum value) of $f(\mathbf{x})$ subject to the constraint $\mathbf{x} \in S$.

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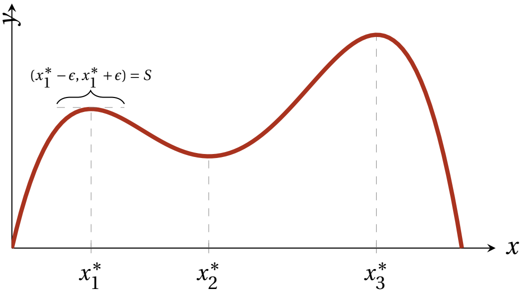

Local maximum around the interval S

PROPOSITION: Let $g(\mathbf{z})$ be a strictly increasing function of a single variable, that is:

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REMARK: This fact is useful since a function $f(\mathbf{x})$ can be transformed in such a way that the resulting function is easier to work with.

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$$v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$$

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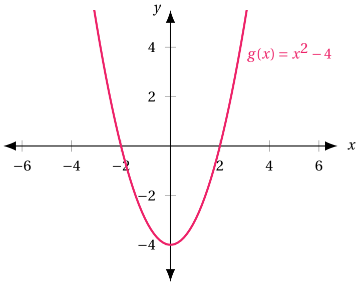
As it turns out that any minimisation problem can be converted into one of maximisation flipping upside down the objective function $f(\mathbf{x})$, so that:

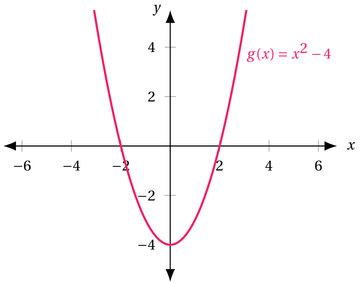
$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}) \\ s.t. & \mathbf{x} \in S \end{array} \quad \text{is equal to} \quad \begin{array}{ll} \max_{\mathbf{x}} & -f(\mathbf{x}) \\ s.t. & \mathbf{x} \in S \end{array}$$

Example:



Minimisation problem

Example:



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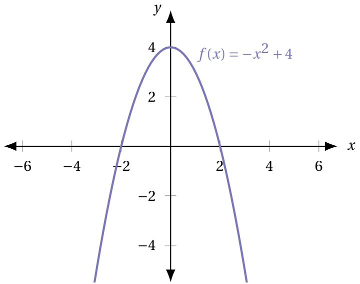

Maximisation problem

EXTREME VALUE THEOREM: let $f(\mathbf{x})$ be a **continuous** function defined on X and let S be a **compact** subset of X. Then the problems:

$$\begin{array}{ll} \min_{\mathbf{x}} & f(\mathbf{x}) \\ s.t. & \mathbf{x} \in S \end{array} \quad \text{and} \quad \begin{array}{ll} \max_{\mathbf{x}} & f(\mathbf{x}) \\ s.t. & \mathbf{x} \in S \end{array}$$

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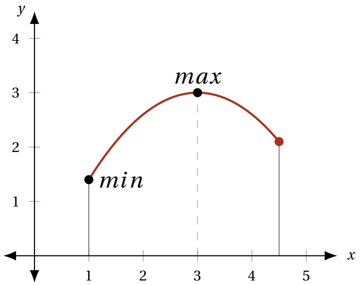

What if the conditions for an optimum are **relaxed**, i.e. are not met?:

BOUNDEDNESS: The set S is bounded if there exists a number $k < \infty$ such that the distance of every point in S from the origin is at most k.

Example:

- Bounded set: $S = \{(x, y) \in \mathbb{R}^2 | 0 < x < 1, -10 \leq y < \pi/2\}$
- Unbounded set: $S = \{(x, y) \in \mathbb{R}^2 | 0 < x < \infty, -10 \leq y < \pi/2\}$

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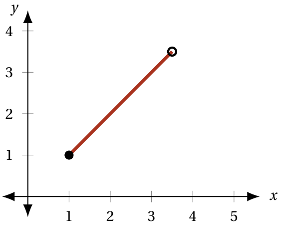

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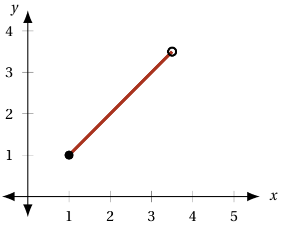

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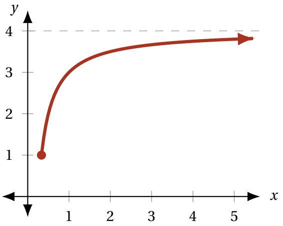

Unbounded

CLOSEDNESS:

- The set S of n-vectors is open if every point in S is an interior point of S.
- The set S of n-vectors is closed if every boundary point of S is a member of S.

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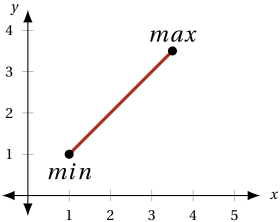

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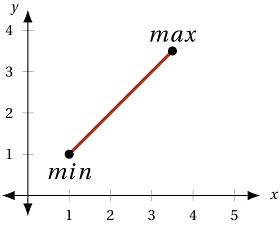
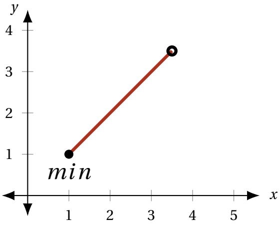


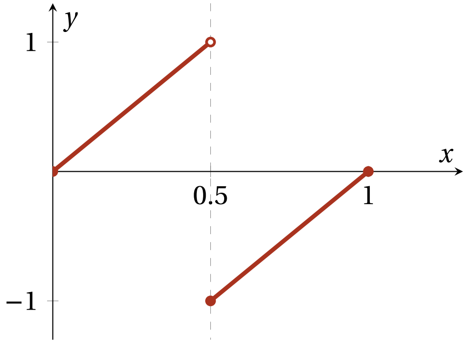
Closed



Open

CONTINUITY: a function is continuous if $\lim_{x \to a} f(x) = f(a)$

Example: relaxing continuity



Non-continuous function

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DEFINITION: Let the function $f(\mathbf{x})$ be defined on a set S. A point $x \in S$ is a **stationary point** of $f(\mathbf{x})$ if $f(\mathbf{x})$ is differentiable and $f_i(\mathbf{x}) = 0$, for $i = 1, 2, \ldots$.

INTERIOR OPTIMA

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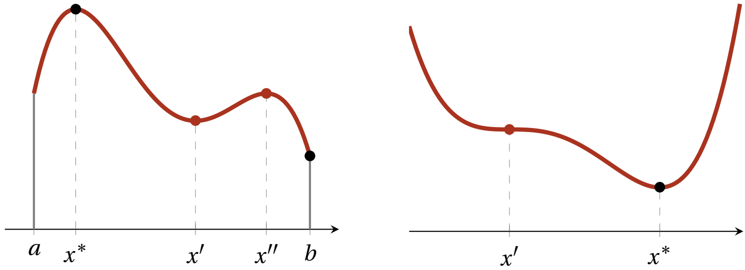

- In the left figure the points x^*, x', x'' are stationary points and extreme points. In the right figure x' is a stationary point but not a extreme
- On the left picture b is a extreme point but is not a stationary point

So, In other words:

1. A stationary point might not be a local maximiser
2. A local maximiser might not be a stationary point

INTERIOR OPTIMA

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Then why is it interesting if at all?

The only case in which a local maximiser is not a stationary point is when it is at the boundary of the set. That is, any **interior point** that is a maximiser must be a stationary point.

INTERIOR OPTIMA

FIRST ORDER CONDITIONS

PROPOSITION: Let $f(\mathbf{x})$ be defined on the set S. If $\mathbf{x}$ is a maximiser in the interior of S and the partial derivatives exist w.r.t. the $i-th$ variable. Then:

$$f_i(x) = 0, \quad \forall i = 1, ..., n$$

This result gives a **necessary condition** for $\mathbf{x}$ to be a maximiser (or a minimiser) of $f(\mathbf{x})$

The condition is obviously **not sufficient** for a point to be a maximiser (could be minimiser or inflexion point)

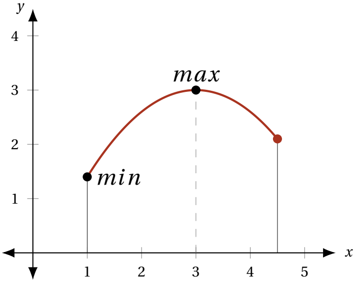
The first-derivative is involved, so we refer to the condition as a **first-order condition FOC**

INTERIOR OPTIMA

FIRST ORDER CONDITIONS

PROOF: Let the point $\mathbf{x}^*$ be a local maximiser, then it is clear that $f(x_1^* + h_1, \mathbf{x}_{-1}) \leq f(x_1^*, \mathbf{x}_{-1})$ for any $(x_1^* + h_1, \mathbf{x}_{-1}) \in S$, or in other words $f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1}) \leq 0$.

- Approaching the point from the right: $h > 0 \Rightarrow \lim_{h^+ \to 0} \frac{f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1})}{h} \leq 0$
- Approaching the point from the left: $h < 0 \Rightarrow \lim_{h^- \to 0} \frac{f(x_1^* + h_1, \mathbf{x}_{-1}) - f(x_1^*, \mathbf{x}_{-1})}{h} \geq 0$



Because the continuity of $f(x)$, there will be a point on I such that $f'(x) = 0$

The previous proposition give us the sufficient conditions for a point to be a stationary point

INTERIOR OPTIMA

FIRST ORDER CONDITIONS

The previous proposition give us the sufficient conditions for a point to be a stationary point

IF:

- $\mathbf{x}^*$ is a maximiser
- x^* is in the interior of S
- f_i exist $\forall i = 1, 2, \ldots$

THEN:

- $\mathbf{x}^*$ is a **Stationary Point**, i.e. $f_i'(\mathbf{x}^*) = 0 \ \forall i = 1, 2, \ldots$

Procedure to solve a maximisation problem

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Let f be a differentiable function of n variables and let S be a set of n-vectors. If the problem

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2. Along them find the values of the function at the boundary of S
3. The largest values of $f(\mathbf{x}^*)$ are the maximisers of f.

Example 1: Consider the problem:

$$\max_{x,y} \quad f(x,y) = -(x-1)^2 - (y+2)^2$$
$$s.t. \quad -\infty < x < \infty,$$
$$-\infty < y < \infty$$

The problem does not meet the conditions of the extreme value theorem —$x, y \in (-\infty, \infty)$ —so it is not possible to know beforehand if the problem will have a solution.

First order conditions:

$$f_x(x,y) = -2(x-1) = 0 \quad \Rightarrow \quad x^* = 1$$
$$f_y(x,y) = -2(y+2) = 0 \quad \Rightarrow \quad y^* = -2$$

Then, the point $(1,-2)$ is stationary, we do not know yet if it is a maximiser.

Example 2: Consider the problem:

$$\begin{aligned} \max_{x,y} \quad & f(x,y) = (x-1)^2 + (y-1)^2 \\ s.t. \quad & 0 \le x \le 2, \\ & -1 \le y \le 3 \end{aligned}$$

The problem does meet the conditions of the extreme value theorem - $x, y \in S$ - so it is possible to know beforehand that the problem will have maximum(a) and minimum(a).

First order conditions:

$$\begin{aligned} f_x(x,y) &= 2(x-1) = 0 \qquad \Rightarrow \qquad x^* = 1 \\ f_y(x,y) &= 2(y-1) = 0 \qquad \Rightarrow \qquad y^* = 1 \end{aligned}$$

Then the point $(x^*, y^*) = (1,1)$ is stationary, where $f(x^*, y^*) = 0$

INTERIOR OPTIMA

FIRST ORDER CONDITIONS

Example 2: Continuation:

Now consider the behaviour of the objective function on the boundary of the set S, which is a rectangle:

- Consider $x = 0$ and $-1 \leq y \leq 3$ then $f(0, y) = 1 + (y-1)^2$. By the FOC: $f_y(0, y^*) = 2(y-1) = 0 \Rightarrow y = 1$ which is in $int(S)$. Again we look at the boundary points in $\{(0, y) \in \mathbb{R}^2 | -1 \leq y \leq 3\}$, i.e. the points $(0, -1)$ and $(0, 3)$ are the candidates for optima where the value of the function is $f(0, -1) = f(0, 3) = 5$
- A similar analysis leads to points $(2, -1)$ and $(2, 3)$ being candidates for optima and where the function attains $f(2, -1) = f(2, 3) = 5$

Comparing the values of the function at the stationary points $(1, 1)$ and at the boundary points $(0, -1), (0, 3), (2, -1)$ and $(2, 3)$ we can conclude that the function has 4 solutions.

Example 3: Consider the problem:

$$\begin{cases} \max_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{cases} \text{and} \begin{cases} \min_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{cases}$$

These problems meet the criteria of the extreme value theorem and hence they have solutions.

FOC:

$$\left.\begin{array}{r} f_x(x,y) = 2x = 0 \Rightarrow x^* = 0 \\ f_y(x,y) = 2y + 1 = 0 \Rightarrow y^* = -\frac{1}{2} \end{array}\right\} \Rightarrow (x^*, y^*) = \left(0, -\frac{1}{2}\right)$$

Then $\left(0, -\frac{1}{2}\right)$ is a stationary point where $f\left(0, -\frac{1}{2}\right) = -\frac{5}{4}$.

Example 3: Continuation

Turning to the boundary points we look at points that lay on the boundary, i.e. $x^2 + y^2 = 1$. Taking this equality into account the problem can be transform:

$$\text{from} \quad \begin{array}{cc} \max_{x,y} & f(x,y) = x^2 + y^2 + y - 1 \\ s.t. & x^2 + y^2 \leq 1 \end{array}$$

$$\text{into} \quad \begin{array}{cc} \max_{y} & f(y) = 1 + y - 1 = y \\ s.t. & 0 \leq y \leq 1 \end{array}$$

Clearly the minimum of this new problem is at $(1,0)$ and the maximum at $(0,1)$ where the functions attain 0 and 1 respectively.

Comparing the stationary and boundary points we see that the maximum is at $(0,1)$ and the minimum at $\left(0, -\frac{1}{2}\right)$

INTERIOR OPTIMA

SECOND ORDER CONDITIONS

MATHEMATICAL DETOUR:

Hessian Matrix: it is the matrix of second derivatives of a function

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_m x_1} & f_{x_m x_2} & \cdots & f_{x_m x_n} \end{pmatrix}$$

INTERIOR OPTIMA

SECOND ORDER CONDITIONS

PROPOSITION: Let $f(\mathbf{x})$ be a twice-differentiable function with continuous partial derivatives and cross partial derivatives, defined on the set S. Suppose that $f_i(\mathbf{x}^*) = 0$, $\forall i$ for some $\mathbf{x}^*$ in the interior of S (so that $\mathbf{x}^*$ is a stationary point of f). Let $\mathbf{H}$ be the Hessian of $f(\mathbf{x})$:

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- If $\mathbf{H}(\mathbf{x}^*)$ is negative definite then $\mathbf{x}^*$ is a local maximiser
- If $\mathbf{x}^*$ is a local maximiser then $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite
- If $\mathbf{H}(\mathbf{x}^*)$ is positive definite then $\mathbf{x}^*$ is a local minimiser
- If $\mathbf{x}^*$ is a local minimiser then $\mathbf{H}(\mathbf{x}^*)$ is positive semi-definite

INTERIOR OPTIMA

SECOND ORDER CONDITIONS

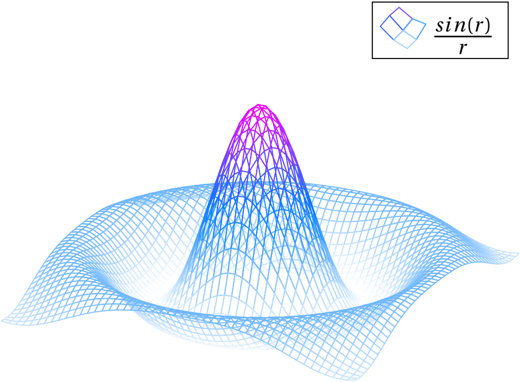
The previous slide implies that:

- If $\mathbf{H}(\mathbf{x}^*)$ is negative semi-definite Then $\mathbf{x}^*$ is either a maximiser or a saddle point
- If $\mathbf{H}(\mathbf{x}^*)$ is positive semi-definite Then $\mathbf{x}^*$ is either a minimiser or a saddle point

For this reason the determinant test should be summoned:

- If $|\mathbf{H}(\mathbf{x}^*)| < 0$ Then $\mathbf{x}^*$ is a saddle point
- If $|\mathbf{H}(\mathbf{x}^*)| > 0$ and $\mathbf{H}(\mathbf{x}^*)$ is n.s.d. Then $\mathbf{x}^*$ is a maximum point
- If $|\mathbf{H}(\mathbf{x}^*)| > 0$ and $\mathbf{H}(\mathbf{x}^*)$ is p.s.d. Then $\mathbf{x}^*$ is a minimum point
- If $|\mathbf{H}(\mathbf{x}^*)| = 0$ Then the test is inclusive. Solve by inspection

Exmple using the mesh parameter



Example 1: Consider the problem:

$$\max_{x,y} f(x,y) = x^3 + y^3 - 3xy$$

FOC:

$$\left.\begin{array}{l} f_x(x,y) = 3x^2 - 3y = 0 \Rightarrow x^2 = y \\ f_y(x,y) = 3y^2 - 3x = 0 \Rightarrow x = y^2 \end{array}\right\} \Rightarrow y = y^4 \text{ then } \left.\begin{array}{l} (x,y) = (0,0) \\ (x,y) = (1,1) \end{array}\right\}$$

INTERIOR OPTIMA

SECOND ORDER CONDITIONS

Example 1:

Now the hessian of $f(x,y)$ at (x,y) is:

$$\mathbf{H}(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Turning to the hessian test:

1. $|\mathbf{H}(0,0)| = -9 < 0$ then is a saddle point
2. $|\mathbf{H}(1,1)| = 27 > 0$ also $f_{xx}(1,1) = 6$ and $f_{yy}(1,1) = 6$ and so the point is a local minimiser

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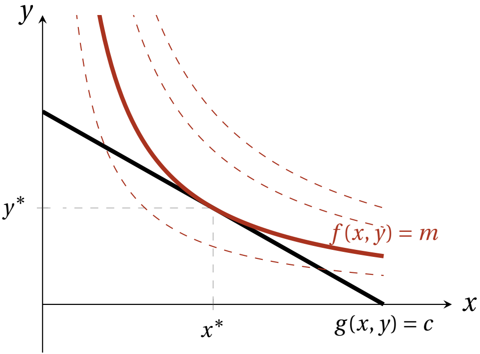
EQUALITY CONSTRAINTS

INTRODUCTION

Example : consider the problem $\max_{x,y} f(x,y) = xy$ s.t. $g(x,y) = c$



PROPOSITION : let $f(x,y)$ and $g(x,y)$ be continuously differentiable functions of two variables defined on the set S, let c be a number, and assume $(\mathbf{x}^*,\mathbf{y}^*)$ is an interior point of S that solves the problem:

$$\begin{array}{cc} \max_{x,y} & f(x,y) \\ s.t. & g(x,y)=c \end{array} \quad \text{or} \quad \begin{array}{cc} \min_{x,y} & f(x,y) \\ s.t. & g(x,y)=c \end{array}$$

Suppose also that either $g_x(x,y) \neq 0$ or $g_y(x,y) \neq 0$.

EQUALITY CONSTRAINTS

NECESSARY CONDITIONS

Then there is a unique number λ such that $(\mathbf{x}^*, \mathbf{y}^*)$ is a stationary point of the **Lagrangian**:

$$\mathcal{L} = f(x, y) - \lambda \left(g(x, y) - c \right)$$

That is, $(\mathbf{x}^*, \mathbf{y}^*)$ satisfies the FOC:

$$\begin{aligned}
\mathcal{L}_x &= f_x(x, y) - \lambda g_x(x, y) = 0 \\
\mathcal{L}_y &= f_y(x, y) - \lambda g_y(x, y) = 0 \\
\mathcal{L}_\lambda &= g(x, y) - c = 0
\end{aligned}$$

Example 1: Consider the problem:

$$\max_{x,y} xy$$
$$s.t.\ x + y = 6$$

Where the objective function xy is defined on the set of all 2-vectors and the set S is a line, so it is not bounded and the extreme value theorem does not apply.

The lagrangean is:

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(x + y - 6)$$

Example 1: Continuation:
FOC are:

$$\mathscr{L}_x(x, y, \lambda) = y - \lambda = 0$$
$$\mathscr{L}_y(x, y, \lambda) = x - \lambda = 0$$
$$\mathscr{L}_\lambda(x, y, \lambda) = x + y = 6$$

These equations have a unique solution $(x^*, y^*, \lambda^*) = (3, 3, 3)$. Also we have $g_x = 1 \neq 0$ and $g_y = 1 \neq 0$, $\forall (x, y)$, so if the problem has a solution it must be at $(3, 3)$

EQUALITY CONSTRAINTS

NECESSARY CONDITIONS

Example 2: Consider the problem:

$$\max_{x,y} x^2 y$$
$$s.t.\ 2x^2 + y^2 = 3$$

Where the objective function xy is defined on the set of all 2-vectors and the set S is compact, so the extreme value theorem guaranties a solution.

The Lagrangian is:

$$\mathscr{L}(x, y, \lambda) = x^2 y - \lambda(2x^2 + y^2 - 3)$$

Example 2: Continuation:
FOC are:

$$\mathscr{L}_x(x,y,\lambda) = 2x(y-2\lambda) = 0 \tag{1}$$

$$\mathscr{L}_y(x,y,\lambda) = x^2 - 2\lambda y = 0 \tag{2}$$

$$\mathscr{L}_\lambda(x,y,\lambda) = 2x^2 + y^2 - 3 = 0 \tag{3}$$

To find the solutions to the system of equations notice that to meet the first equation either $x = 0$ or $y = 2\lambda$.

Example 2: Continuation:

In turns:

- If $x = 0$, then (3) implies $y = \pm\sqrt{3}$ and (2) resulst in $\lambda = 0$.
- If $y = 2\lambda$, plugging it into (2): $x^2 - y^2 = 0 \Leftrightarrow x^2 = y^2 \Leftrightarrow x = \pm y$
    - If $x = y$, plugging this into (3) results in $3x^2 = 3 \Leftrightarrow x = \mp 1$ and as a result $y = \pm 1$
    - If $x = -y$, plugging this into (3) results in $3x^2 = 3 \Leftrightarrow x = \pm 1$ and as a result $y = \mp 1$

Then the possible **solutions** are:

$$(0, \sqrt{3}, 0) \text{ with } f(0, \sqrt{3}) = 0 \qquad (0, -\sqrt{3}, 0) \text{ with } f(0, -\sqrt{3}) = 0$$

$$\left(1, 1, \frac{1}{2}\right) \text{ with } f(1,1) = 1 \qquad \left(-1, -1, -\frac{1}{2}\right) \text{ with } f(-1,-1) = -1$$

$$\left(1, -1, -\frac{1}{2}\right) \text{ with } f(1,-1) = -1 \qquad \left(-1, 1, \frac{1}{2}\right) \text{ with } f(-1,1) = 1$$

Example 2: Continuation:

Now $g_x = 4x$ and $g_y = 2y$, the only value in which $g_x = g_y = 0$ is $(0,0)$. At this point the constraint is not satisfied, thus the only solutions are the ones that meet the FOC.

Since it is a maximisation problem we can safely conclude that the only solution is $(x,y) = (1,1)$ and $(x,y) = (-1,1)$

INTUITION: the value of the **Lagrange multiplier** at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed.

Example: Consider the problem

$$\max_x x^2$$
$$s.t. x = c$$

The solution of this problem is obvious: $x = c$. The maximised value of the function is thus c^2, so that the derivative of this maximised value with respect to c is $2c$.

Let's check that the value of the Lagrange multiplier at the solution of the problem is equal to $2c$. The Lagrangian is:

$$\mathscr{L}(x) = x^2 - \lambda(x - c)$$

so the first-order condition is

$$2x - \lambda = 0$$

The constraint is $x = c$, so the pair (x, λ) that satisfies the first-order condition and the constraint is $(c, 2c)$. Thus we see that indeed λ is equal to the derivative of the maximised value of the function with respect to c.

DEFINITION: the determinant $\mathbf{D}\left(x^*, y^*, \lambda^*\right)$ is called the **Bordered Hessian of the Lagrangian** and takes the following form:

$$\mathbf{D}\left(x^*, y^*, \lambda^*\right) = \begin{vmatrix} \mathscr{L}_{\lambda\lambda} & \mathscr{L}_{\lambda x} & \mathscr{L}_{\lambda y} \\ \mathscr{L}_{x\lambda} & \mathscr{L}_{xx} & \mathscr{L}_{xy} \\ \mathscr{L}_{y\lambda} & \mathscr{L}_{yx} & \mathscr{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{vmatrix}$$

With this in mind we can state the following result

EQUALITY CONSTRAINTS

SUFFICIENT CONDITIONS

PROPOSITION: Let $f(x, y)$ and $g(x, y)$ be twice differentiable functions of two variables defined on the set S and let c be a number. Suppose that $(x*, y*)$, an interior point of S, and the number λ^* satisfy the first-order conditions:

$$
\begin{aligned}
f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) &= 0 \\
f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) &= 0 \\
g(x^*, y^*) &= c
\end{aligned}
$$

Then:

- If $\mathbf{D}\left(x^*, y^*, \lambda^*\right) > 0$ then (x^*, y^*) is a local maximiser
- If $\mathbf{D}\left(x^*, y^*, \lambda^*\right) < 0$ then (x^*, y^*) is a local minimiser

Example: Continuation of the previous one in "NECESSARY CONDITIONS":

The possible solutions where worked out:

$$(0,\sqrt{3},0) \text{ with } f(0,\sqrt{3})=0 \qquad (0,-\sqrt{3},0) \text{ with } f(0,-\sqrt{3})=0$$
$$\left(1,1,\frac{1}{2}\right) \text{ with } f(1,1)=1 \qquad \left(-1,-1,-\frac{1}{2}\right) \text{ with } f(-1,-1)=-1$$
$$\left(1,-1,-\frac{1}{2}\right) \text{ with } f(1,-1)=-1 \qquad \left(-1,1,\frac{1}{2}\right) \text{ with } f(-1,1)=1$$

It seems obvious that the points $(1,1)$ and $(-1,1)$ where global maximisers and the points $(1,-1)$ and $(-1,-1)$ where global minimisers.

But what about $(0,\sqrt{3})$ and $(0,-\sqrt{3})$? They are neither optima, are they local optima?

Example: Continuation

The determinant of the bordered hessian of the Lagrangian is in general:

$$\mathbf{D}\left(x,y,\lambda\right)=\begin{vmatrix}0 & 4x & 2y\\ 4x & 2y-4\lambda & 2x\\ 2y & 2x & -2\lambda\end{vmatrix}=8\left[2\lambda\left(2x^2+y^2\right)+y\left(4x^2-y^2\right)\right]$$

And at the solutions:

- $\left|D(0,\sqrt{3},0)\right|=-8\cdot 3^{\frac{3}{2}}$, and then $(0,\sqrt{3},0)$ is a local minimiser
- $\left|D(0,-\sqrt{3},0)\right|=8\cdot 3^{\frac{3}{2}}$, and then $(0,-\sqrt{3},0)$ is a local maximiser

EQUALITY CONSTRAINTS

n VARIABLES AND m CONSTRAINTS

The Lagrangian method can easily be generalised to a problem of the form:

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

where $\mathbf{x} = (x_1, \ldots, x_n)$.

Ending with a problem of n variables and m constraints.

The Lagrangean for this problem is:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j \right)$$

That is, there is one Lagrange multiplier for each constraint.

DEFINITION: For $j = 1,...,m$ let $g_j(\mathbf{x})$ be a differentiable function of n variables. The **Jacobian Matrix** of $(g_1,...,g_m)$ at the point x is:

$$\begin{pmatrix} g_{1x_1}(\mathbf{x}) & ... & g_{1x_n}(\mathbf{x}) \\ ... & ... & ... \\ g_{mx_1}(\mathbf{x}) & ... & g_{mx_n}(\mathbf{x}) \end{pmatrix}$$

EQUALITY CONSTRAINTS

n VARIABLES AND m CONSTRAINTS

PROPOSITION: Let $f(\mathbf{x})$ and $g_j(\mathbf{x}) = c_j$ for $j = 1, \ldots, m$ be continuously differentiable functions of n variables defined on the set S, with $m \leq n$, let c_j for $j = 1, \ldots, m$ be numbers, and suppose that $\mathbf{x}^*$ is an interior point of S that solves the problem:

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

or the problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$
$$s.t.\ g_j(\mathbf{x}) = c_j \text{ for } j = 1, \ldots, m$$

Suppose also that the rank of the Jacobian matrix of $(g_1, \ldots, g_m)$ at the point x^* is m.

EQUALITY CONSTRAINTS

n VARIABLES AND m CONSTRAINTS

Then there exist unique numbers $\lambda_1, ..., \lambda_m$ such that x^* is a stationary point of the Lagrangian function L defined by:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j\right)$$

That is, $\mathbf{x}^*$ satisfies the FOC:

$$\mathscr{L}_i(\mathbf{x}) = f_i(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j g_{ji}(\mathbf{x}) = 0 \text{ for } i = 1, ..., n$$

In addition, $g_j(\mathbf{x}^*) = c_j$ for $j = 1, ..., m$

EQUALITY CONSTRAINTS

n VARIABLES AND m CONSTRAINTS

Example: Consider the problem:

$$\min_{x,y,z} x^2 + y^2 + z^2$$
$$s.t.\ x + 2y + z = 1$$
$$2x - y - 3z = 4$$

The Lagrangian is:

$$\mathscr{L}(x, y, z) = x^2 + y^2 + z^2 - \lambda_1 \left(x + 2y + z - 1\right) - \lambda_2 \left(2x - y - 3z - 4\right)$$

This function is convex for any values of λ_1 and λ_2, so that any interior stationary point is a solution of the problem. Further, the rank of the Jacobian matrix is 2 (a fact you can take as given), so any solution of the problem is a stationary point. Thus the set of solutions of the problem coincides with the set of stationary points.

EQUALITY CONSTRAINTS

n VARIABLES AND m CONSTRAINTS

Example: Continuation:

FOC are:

$$2x - \lambda_1 - 2\lambda_2 = 0 \tag{1}$$
$$2y - 2\lambda_1 + \lambda_2 = 0 \tag{2}$$
$$2z - \lambda_1 + 3\lambda_2 = 0 \tag{3}$$
$$x + 2y + z = 1 \tag{4}$$
$$2x - y - 3z = 4 \tag{5}$$

Solving (1) and (2) for λ_1 and λ_2 gives:

$$\lambda_1 = \frac{2}{5}x + \frac{4}{5}y \tag{6}$$
$$\lambda_2 = \frac{4}{5}x + \frac{2}{5}y \tag{7}$$

Example: Continuation:

Now substitute (6) and (7) into (3) and solve the system of equations:

$$x = \frac{16}{15},\ y = \frac{1}{3},\ z = -\frac{11}{15},\ \lambda^1 = \frac{52}{75} \text{ and } \lambda^2 = \frac{54}{75}$$

Then we can conclude that $(x, y, z) = \left(\frac{15}{16}, \frac{1}{3}, -\frac{11}{15}\right)$ is the unique solution to the problem.

EQUALITY CONSTRAINTS

ENVELOPE THEOREM

PROPOSITION: Let $f(\mathbf{x};\mathbf{r})$ be a function of n variables, let $\mathbf{r}$ be a h-vector of parameters, and let the n-vector $\mathbf{x}^*$ be a maximiser of $f(\mathbf{x};\mathbf{r})$. Assume that the partial derivative $f'_{n+k}(\mathbf{x}^*,\mathbf{r})$ (i.e. the partial derivative of $f(\mathbf{x};\mathbf{r})$ with respect to $\mathbf{r}_k$) at (x^*,r) exists. Define the **Value Function** $f^*(\mathbf{r})$ of k variables by:

$$f^*(\mathbf{r}) = \max_x f(\mathbf{x};\mathbf{r}), \quad \forall r_k.$$

If the partial derivative $f_k^*(\mathbf{r})$ exists then

$$f_k^*(\mathbf{r}) = f_{n+k}(\mathbf{x}^*,\mathbf{r}).$$

For $k = \{1,...,h\}$

INTUITION: we might be interested in seeing how the function at the solution $f(\mathbf{x}^*;\mathbf{r})$ changes as some parameters $\mathbf{r}$ change.

RESULT: At the optimum only direct effects of the parameters into the function need taking into account, the indirect effects can be neglected since:

$$\frac{\partial f(\mathbf{x}^*(\mathbf{r});\mathbf{r})}{\partial r_k} = \frac{\partial f(\mathbf{x}^*(\mathbf{r});\mathbf{r})}{\partial x_i^*(\mathbf{r})} \cdot \frac{\partial x_i^*(\mathbf{r})}{\partial r_k} + \frac{\partial f^*(\mathbf{r})}{\partial r_k}$$

But at the optimum $\frac{\partial f(\mathbf{x}^*;\mathbf{r})}{\partial x_i^*} = 0$

Hence the result

Example: Consider the following function $f(x;\mathbf{r}) = x^{r_1} - r_2 x$ where $0 < r_1 < 1$. Which has a maximisation point at:

$$x^* = \left(\frac{r_1}{r_2}\right)^{\frac{1}{1-r_1}}$$

It might be interesting to know the effect of r_1 in the change of the value function. Thus by the envelope theorem:

$$\frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_1} = \left(x^*(\mathbf{r})\right)^{r_1} \ln x^*(r)$$

or substituting $x^*(\mathbf{r})$

$$\frac{\partial f\left(x^*(\mathbf{r});\mathbf{r}\right)}{\partial r_1} = \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{1-r_1}} \frac{1}{1-r_1} \ln\left(\frac{r_1}{r_2}\right)$$

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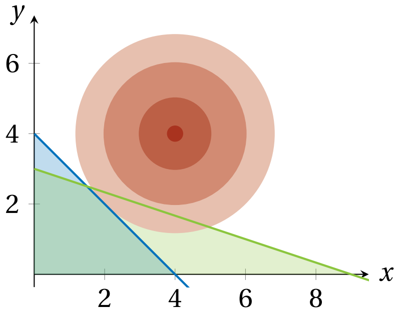
INEQUALITY CONSTRAINTS

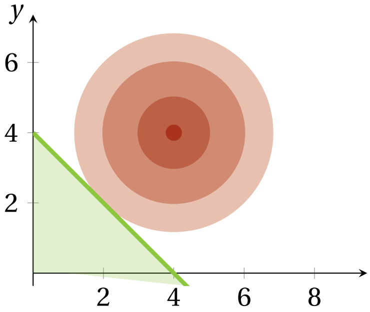
INTRODUCTION

Example :
Consider the problem

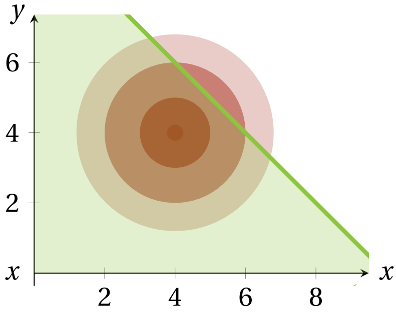
$$\max_{x,y} f(x,y)$$
$$s.t. g_1(x,y) - c_1 \leq 0$$
$$g_2(x,y) - c_2 \leq 0$$
$$x \geq 0, y \geq 0$$



Examples:



The constrained is binding



The constrained is not binding

DEFINITION: let $f(\mathbf{x})$ and $g_j(\mathbf{x})$ be differentiable functions of n variables and let c_j for $j = 1, \ldots, m$ be numbers. Also define the function $\mathscr{L}$ of n variables as:

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j \right) \text{ for all } \mathbf{x}$$

INEQUALITY CONSTRAINTS

KUHN-TUCKER CONDITIONS

The **Kuhn-Tucker conditions** of the problem:

$$\max_{\mathbf{x}} f(\mathbf{x}),\ s.t.\ g_j(\mathbf{x}) - c_j \leq 0 \text{ for } j = 1, \ldots, m$$

are:

- $\mathscr{L}_i(\mathbf{x}) = 0$ for $i = 1, \ldots, n$
- $\lambda_j \left[g_j(\mathbf{x}) - c_j \right] = 0$ for $j = 1, \ldots, n$
- $\lambda_j \geq 0$
- $g_j(\mathbf{x}) \leq c_j$

The **SOLVING PROBLEM RECIPE**: consider the following problem:

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$s.t. \, g_j(\mathbf{x}) \leq c_j \text{ for } j = 1, \ldots, m$$

Where $\mathbf{x} = (x_1, \ldots, x_n)$

INEQUALITY CONSTRAINTS

KUHN-TUCKER CONDITIONS

STEP 1: Write down the Lagrangian

$$\mathscr{L}(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \left(g_j(\mathbf{x}) - c_j\right)$$

With $\lambda_1, \ldots, \lambda_m$ as the Lagrange multipliers with the m constraints

STEP 2: Equate all the first-order partial derivatives of $\mathscr{L}(\mathbf{x})$ to 0:

$$\frac{\partial \mathscr{L}(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

STEP 3: Impose the complementary slackness conditions:

$$\lambda_j \left[g_j(\mathbf{x}) - c_j \right] = 0, \quad j = 1, \ldots, m$$

where either $\lambda_j > 0$ or $\lambda_j = 0$

STEP 4: Require $\mathbf{x}$ to satisfy the constraints:

$$g_j(\mathbf{x}) \leq c_j$$

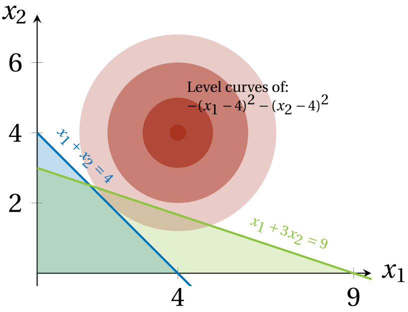
Example :

Consider the problem

$$\max_{x_1, x_2} -(x_1-4)^2-(x_2-4)^2$$

$$s.t. x_1+x_2 \leq 4$$

$$x_1+3x_2 \leq 9$$



INEQUALITY CONSTRAINTS

KUHN-TUCKER CONDITIONS

STEP 1: Write down the Lagrangian

$$\mathscr{L}(x_1, x_2) = -(x_1 - 4)^2 - (x_2 - 4)^2 - \lambda_1 (x_1 + x_2 - 4) - \lambda_2 (x_1 + 3x_2 - 9)$$

STEP 2: Equate all the first-order partial derivatives of $\mathscr{L}(\mathbf{x})$ to 0:

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_1} = -2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_2} = -2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$
$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial \lambda_1} = x_1 + x_2 - 4 = 0$$
$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial \lambda_2} = x_1 + 3x_2 - 9 = 0$$

STEP 3: Impose the complementary slackness conditions, in other words try the following four cases:

1. $\lambda_1 = \lambda_2 = 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 < 9$
2. $\lambda_1 > 0$ and $\lambda_2 = 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 < 9$
3. $\lambda_1 = 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 = 9$
4. $\lambda_1 > 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 = 9$

INEQUALITY CONSTRAINTS

KUHN-TUCKER CONDITIONS

CASE 1: $\lambda_1 = \lambda_2 = 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 < 9$, None of the constraints are binding and the FOC become:

$$\left.\begin{aligned} \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_1} &= -2(x_1 - 4) = 0 \\ \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_2} &= -2(x_2 - 4) = 0 \end{aligned}\right\} \Rightarrow (x_1^*, x_2^*) = (4, 4)$$

BUT introducing these values into the constrain $x_1 + x_2 \leq 4$:

$$4 + 4 \leq 4 \ \lightning$$

Hence arriving to a contradiction and being able to discard $(4, 4)$

INEQUALITY CONSTRAINTS

KUHN-TUCKER CONDITIONS

CASE 2: $\lambda_1 > 0$ and $\lambda_2 = 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 < 9$, The first constraint is binding but the sencond is not, the FOC become:

$$\left.\begin{aligned} \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_1} &= -2(x_1-4)-\lambda_1 = 0 \\ \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_2} &= -2(x_2-4)-\lambda_1 = 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 \quad (1)$$

$$\frac{\partial \mathscr{L}(x_1,x_2)}{\partial \lambda_1} = x_1 + x_2 - 4 = 0 \quad (2)$$

Plugging (1) into (2)

$$x_1 + x_1 = 4 \Rightarrow x_1^* = 2, x_2^* = 2$$

Checking the result against the other constraint $x_1 + 3x_2 \leq 9$:

$$2 + 3 \cdot 2 = 8 \leq 9$$

And then the point (2,2) is a candidate for a solution

CASE 3: $\lambda_1 = 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 < 4$ and $x_1 + 3x_2 = 9$, The first constraint is not binding but the sencond is, the FOC become:

$$\left.\begin{array}{lll} \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_1} & = -2(x_1-4) - \lambda_2 = 0 & \Rightarrow \lambda_2 = -2(x_1-4) \\ \frac{\partial \mathscr{L}(x_1,x_2)}{\partial x_2} & = -2(x_2-4) - 3\lambda_2 = 0 & \Rightarrow \lambda_2 = -\frac{2}{3}(x_2-4) \end{array}\right\}$$

$$\Rightarrow x_1 = \frac{2}{3}x_2 - \frac{8}{3} \quad (1)$$

$$\frac{\partial \mathscr{L}(x_1,x_2)}{\partial \lambda_1} = x_1 + 3x_2 - 9 = 0 \quad (2)$$

Plugging (1) into (2)

$$\frac{2}{3}x_2 - \frac{8}{3} + 3x_2 = 9 \Rightarrow \frac{10}{3}x_2 = \frac{19}{3} \Rightarrow x_2^* = \frac{19}{10};\; x_1^* = \frac{33}{10}$$

Checking the result against the other constraint $x_1 + x_2 \leq 4$:

$$\frac{33}{10} + \frac{19}{10} = \frac{52}{10} \leq 4 \;\lightning$$

Hence arriving to a contradiction and being able to discard $\left(\frac{33}{10}, \frac{19}{10}\right)$

CASE 4: $\lambda_1 > 0$ and $\lambda_2 > 0$ which implies $x_1 + x_2 = 4$ and $x_1 + 3x_2 = 9$, now both constraints are binding, the FOC become:

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_1} = -2(x_1 - 4) - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial x_2} = -2(x_2 - 4) - \lambda_1 - 3\lambda_2 = 0$$

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial \lambda_1} = x_1 + x_2 - 4 = 0$$

$$\frac{\partial \mathscr{L}(x_1, x_2)}{\partial \lambda_2} = x_1 + 3x_2 - 9 = 0$$

Solving the last two equations:

$$\left.\begin{aligned} x_1 + x_2 &= 4 \\ x_1 + 3x_2 &= 9 \end{aligned}\right\} \Rightarrow (x_1^*, x_2^*) = \left(\frac{3}{2}, \frac{5}{2}\right)$$

Then the first two equations become:

$$\left.\begin{aligned} 5 - \lambda_1 - \lambda_2 &= 0 \\ 3 - \lambda_1 - 3\lambda_2 &= 0 \end{aligned}\right\} \Rightarrow \lambda_1 = 6 \quad \text{and} \quad \lambda_2 = -1 \geq 0 \ ↯$$

Hence arriving to a contradiction and being able discard $\left(\frac{3}{2}, \frac{5}{2}\right)$

SOLUTION: so $(x_1, x_2, \lambda_1, \lambda_2) = (2, 2, 4, 0)$ is the single solution of the Kuhn-Tucker conditions. Hence the unique solution of the problem is $(x_1, x_2) = (2, 2)$