Basic Algebra

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THE EQUATION

- **Variable**: it is an element representing an unknown value.
  Example: $x, y, z, \ldots$ are common ones
- **Expression**: it is a statement about of the value of something.
  Example: $x = 5$
- **Equation**: An equation is a statement that two expressions are equal.
  - **True equation**: when the statement of the equation is true
    Example: $7 = 3 + 4$
  - **False equation**: when the statement of the equation is false
    Example: $7 = 5$
  - **Algebraic equation**: if the equation involves dealing with a variable
    Example: $x + 4 = 7$ or $\frac{4}{x} = 2$

Warning: Division into 0 is not defined.

THE EQUATION

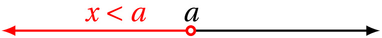
Inequality: it is a statement about if quantity is bigger(or smaller) than other.

- more or equal to: $x \geq a$

- (strictly) more than: $x > a$

- less or equal to: $x \leq a$

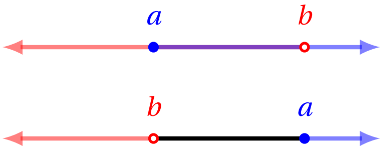
- (strictly) less than: $x < a$



THE EQUATION

Warning: when multiplying/dividing by a negative number the direction of inequality changes.

Compound inequality: an inequality that combines two simple inequalities. OR and AND inequalities. They can have 1, infinitely many or none solution.



Combine both with an OR or an AND.

SET THEORY

DEFINITION: A set is a collection of of objects which are called elements or members of the set

Examples:
Set of vegetables

$$V = \{\text{Aubergine}, \text{Couguette}, \text{Pepper}, \ldots\}$$

Set of AC⚡DC members

$$⚡ACDC⚡ = \{\text{Bon Scott, Angus Young}, \ldots\}$$

However some sets cannot be listed a the ones previously shown, because they might have infinitely many elements. Consequently some times sets have to be defined according to a property.

SET THEORY

Example:
The budget set: The consumer is constrained in the quantity of goods that can buy, according to the prices and her wealth, call:

- x_1: grams of cured ham with price p_{x_1}
- x_2: kilometers of public tranport with price p_{x_2}
- Wealth: w euros

Then the set of bundles that the consumer can buy feasibly is:

$$\mathscr{B} = \left\{(x_1, x_2) \in \mathbb{R} : p_{x_1} x_1 + p_{x_2} x_2 = w,\ x_1 \geq 0,\ x_2 \geq 0\right\}$$

NOTATION:

$$S = \{\text{Typical member: defining properties}\}$$

SET THEORY

SET MEMBERSHIP:

To indicate that an element x **belongs to a set** S, we write:

$$x \in S$$

And we write:

$$x \notin S$$

To indicate that it does not.

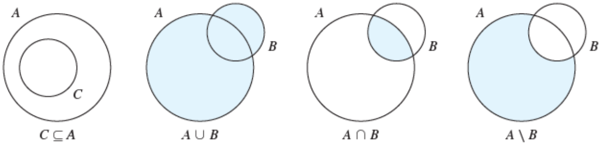
Also it might be interesting to note that a set Q is a **subset** of S

$$Q \subseteq S$$

SET THEORY

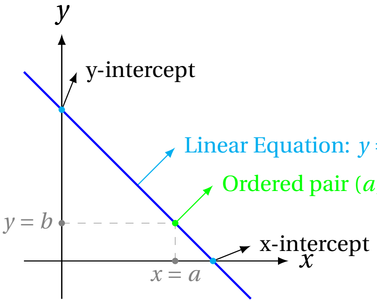
SET OPERATIONS:

- A union B: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- A intersection B: $A \cap B = \{x : x \in A\ ,\ x \in B\}$
- A minus B: $A \setminus B = \{x : x \in A, x \notin B\}$



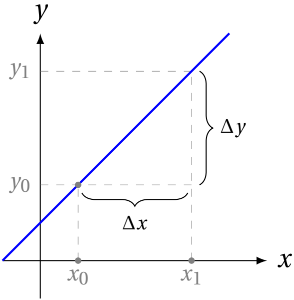
THE LINEAR EQUATION

Linear equation: equation which graph is a line.



THE LINEAR EQUATION

- **Slope**: it is a measure of the steepness of a line.



$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

  - **Point-Slope form:** $\left(y - b\right) = m\left(x - a\right),\ m, a, b \in \mathbb{R}$
  - Slope-intercept form: $y = mx + b,\ m, b \in \mathbb{R}$.
  - Standard form: $ax + by = c,\ a, b, c \in \mathbb{R}$

SYSTEMS OF EQUATIONS

- **Systems of equations**: A System of Equations is when we have two or more equations working together. For the equations to "work together" they share one or more variables.
- **Equivalent systems**: systems with the same solutions.

SYSTEM OF EQUATIONS

Calculation

- Elimination: Find a linear combination of the two equations and subtract one from the other to solve for the variable:

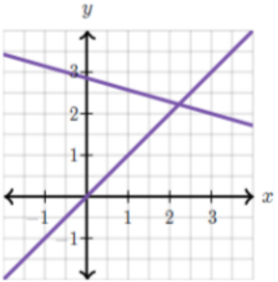
$$\begin{array}{r} ax + by = c \\ \underline{ax + dy = e} \\ (b-d)\,y = c - e \end{array}$$

- Substitution: solve for one variable in one equation and substitute the result into the other(s).

SYSTEM OF EQUATIONS

Solutions

- **Consistent**: a system of equations is consistent if it has at least one solution (it can have infinite solutions).
    - **Independent**: a system of linear equations is independent if it has one solution.

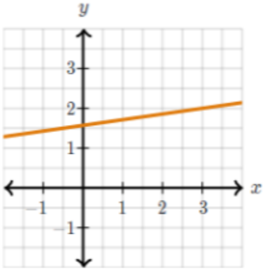


One solution. A system of linear equations has one solution when the graphs intersect at a point.

SYSTEM OF EQUATIONS

Solutions

- **Consistent**: a system of equations is consistent if it has at least one solution.
  - **Dependent**: a system of linear equations is dependent if it has infinitely many solutions.

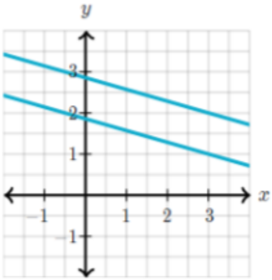


Infinite solutions. A system of linear equations has infinite solutions when the graphs are the exact same line.

SYSTEM OF EQUATIONS

Solutions

- **Inconsistent**: a system of equations is inconsistent if it has no solution.



No solution. A system of linear equations has no solution when the graphs are parallel.

FUNCTIONS

Function: it is a special relationship where each input has a single unique output. It is often written as "$f(x)$" where x is the input value.

- **Interval**: It is a set of bounded numbers.
- **Types of intervals**:
  - $Closed: \{x \in \mathbb{R} | a \leq x \leq b\}$ or $x \in [a, b]$
  - $Open: \{x \in \mathbb{R} | a < x < b\}$ or $x \in (a, b)$
  - $Right-semiclosed: \{x \in \mathbb{R} | a < x \leq b\}$ or $x \in (a, b]$
  - $Left-semiclosed: \{x \in \mathbb{R} | a \leq x < b\}$ or $x \in [a, b)$
  - Unbounded interval: $\{x \in \mathbb{R} | \ a \leq x \leq \infty\}$

- **Domain**: the set of numbers (inputs) for which the function has defined outputs.
- **Co-domain**: the set of values that could possibly come out. The Co-domain is actually part of the definition of the function.
- **Range**: the set of actual values taken by the function

- **Injective (into):** it means that every member of Y is matched by a unique member of X. So if $x \neq y$ then $f(x) \neq f(y)$
- **Surjective (onto):** means that every Y has at least one matching X (maybe more than one). $f(X) = Y$
- **Bijective (One-to-one correspondence):** Bijective means having both properties, Injective and Surjective, together. This property is important because it means that the **function is invertible**

NOTE: These are properties that functions have or do not have, it is not a way of classifying functions.

FUNCTIONS

CHARACTERISTICS

- Extrema: maxima and minima considered collectively
- Local maximum (minimum): The height of the function at a point is greater (smaller) than the height anywhere else in that interval around a.
- Global maximum (minimum): The maximum or minimum over the entire function is called an "Absolute" or "Global" maximum or minimum

FUNCTIONS

CHARACTERISTICS

- Increasing (decreasing) functions: a function is increasing if y increases (decreases) as x increases.
- positive (negative) functions: a function is positive y it is above the horizontal axis, i.e. if $f(x)$ is positive (negative).

Average Rate of Change: it is the change of the function per unit of a variable the interval $x \in (a, b)$.

$$ARC = \frac{f(b) - f(a)}{b - a}$$

FUNCTIONS

OPERATIONS

Combination of functions: As with addition, multiplication, power and exponent of numbers, there can be the same operations defined on functions:

- $\left(f+g\right)(x) = f(x) + g(x)$
- $\left(f-g\right)(x) = f(x) - g(x)$
- $\left(f \cdot g\right)(x) = f(x) \cdot g(x)$
- $\left(f/g\right)(x) = f(x)/g(x)$

Warning: the function that is dividing cannot be 0

The domain of the new function will have the restrictions of **both** functions that made it.

Example: `https://www.desmos.com/calculator/c1wsn98eoz`

FUNCTIONS

OPERATIONS

Composition of functions: It is applying one function to the results of another.

- $(g \circ f)(x) = g(f(x))$, first apply f(), then apply g()
- You must also respect the domain of the first function
- Some functions can be de-composed into two (or more) simpler functions.

Example:

$$\left.\begin{aligned} f(x) &= e^x \\ g(x) &= x^2+1 \end{aligned}\right\} \Rightarrow \begin{aligned} f \circ g = e^{x^2+1} \\ g \circ f = e^{2x}+1 \end{aligned}$$

FUNCTIONS

OPERATIONS

Shifting functions: A function can be shifted up/down or right/left adding/subtracting a number outside or inside the function respectively.

- Horizontally: $g(x) = f(x + a)$
- Vertically: $g(x) = f(x) + a$

Stretching functions: A function can be stretched vertically or horizontally by multiplying/dividing a number outside or inside the function respectively.

- Horizontally: $g(x) = f(x \cdot a)$
- Vertically: $g(x) = f(x) \cdot a$

FUNCTIONS

OPERATIONS

Reflecting functions: a function can be flipped over the x/y axises by multiplying times -1 outside or inside the function respectively.

- x-axis: $g(x) = -f(x)$
- y-axis: $g(x) = f(-x)$

FUNCTIONS

INVERSES

Inverse functions: is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa. i.e., $f(x) = y \Leftrightarrow g(y) = x$

- Back to the original value: it gets to the point where we started $f\left(f(x)\right)^{-1} = x$
- Symmetry: the inverse function is symmetric across the y=x line.
- Calculus: solve for x.

FUNCTIONS

INVERSES

- Solvability: Sometimes it is not possible to find the inverse of a function because the function cannot be solved for x.
- Invertibility: if each output has a unique input then the function is invertible. the horizontal line test.
- Not invertible: when for one value of y there exists two of x, the function is **not invertible**
- Domain: some functions do not have and inverse but it can be fixed restricting the domain.
- Notation: Inverse: $f^{-1}(x)$, reciprocal: $f(x)^{-1} = \frac{1}{f(x)}$, so $f^{-1}(x) \neq f(x)^{-1}$

EXPONENCTIAL

- **Exponential Function:** it is a function of the form:

$$f(x) = b^{tx}$$

  - x: Exponent
  - t: Periods (complete)
  - b: Common ratio/base, is the rate of change of the function.

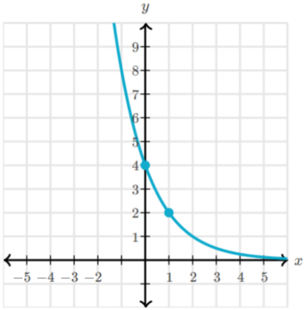
waning: b cannot be negative

- **Exponential growth and decay:**
  - Exponential decay: $0 < b < 1$
  - Exponential growth: $b > 1$

EXPONENCTIAL

Graph:

- Asymptote: Exponential functions have a horizontal asymptote in 0, which can be moved up or down doing suitable transformations.
- Domain: $\mathbb{R}$
- Range: $\mathbb{R}_+ : \ (0, \infty)$



LOGARITHM FUNCTION

BASICS

- **Logarithm Function:** it is a function of the form:

$$f(x) = log_b(x)$$

  - b: base
  - x: power/argument

- **Restrictions:**
  - $b > 0$: In an exponential function, the base b is always defined to be positive
  - $b \neq 1$: Assume $b = 1$, then $1^b = x$, which is not true for any value of $x \neq 1$
  - $x > 0$: Any positive number to any number is positive

Name	Base	Regular Notation	Special Notation
Common	10	$\log_{10}(x)$	$\log(x)$
Natural	e	$\log_e(x)$	$\ln(x)$

LOGARITHM FUNCTION

PROPERTIES

$$M = b^x \Leftrightarrow log_b M = x,\; N = b^y \Leftrightarrow log_b N = y \text{ and } a = b^O \Leftrightarrow log_b a = O$$

- Product Rule: $\log_b M \cdot N = \log_b M + \log_b N$.

$$\log_b(M \cdot N) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y = \log_b M + \log_b N$$

- Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

$$\log_b\left(\frac{M}{N}\right) = \log_b\left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y = \log_b M - \log_b N$$

- Power Rule: $\log_b(M^p) = p \cdot \log_b M$

$$\log_b(M^p) = \log_b(\overbrace{M \cdot \ldots \cdot M}^{p \text{ times}}) = p \cdot \log_b M$$

- Change of Base Rule: $\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$

$$\log_b(a) = 0 \Longleftrightarrow b^0 = a \Longrightarrow \log_x b^0 = \log_x a \Longrightarrow 0 \log_x b = \log_x a \Longrightarrow 0 = \frac{\log_x(a)}{\log_x(b)}$$

LOGARITHM FUNCTION

GRAPH

- **Graph:**
  - Asymptote: Exponential functions have a vertical asymptote in 0, which can be moved left or right doing suitable transformations.
  - Domain: $\mathbb{R}_+ : (0, \infty)$
  - Range: $\mathbb{R}$

LOGARITHM vs EXPONENTIAL

Relation logarithm vs exponential: One is the inverse function of the other

$$a = b^x \Leftrightarrow \log_b(a) = x$$

https://www.khanacademy.org/math/algebra-home/alg-exp-and-log/alg-logarithmic-scale/v/logarithmic-scale

SEQUENCE

BASICS

Sequence: it is an ordered list of numbers

- Terms: each number of the sequence
- Pattern: a rule that tells the following number of the list (if it exists)
- Function: sequences are functions with the caveat $n \in \mathbb{N}$
- Notation: $a(n) = a_n$

SEQUENCE

ARITHMETIC

Arithmetic Sequence: a sequence in which the following number is the addition/subtraction of the previous one.

- Common difference: it is the constant difference between consecutive terms
- Recursive formula:

$$\begin{cases} a(1) = a_1 \\ a(n) = a(n-1) + b \end{cases}$$

- Explicit formula: $a(n) = a(1) + b(n-1)$

SEQUENCE

GEOMETRIC

Geometric Sequence: it is an array of numbers in which each term in the sequence is a fix multiple of the term before.

- Common ratio: The constant ratio between two different terms
- Recursive formula:

$$\begin{cases} a(1) = a_1 \\ a(n) = b \cdot a(n-1) \end{cases}$$

- Explicit formula: $a_n = a_1 \cdot b^{n-1}$

SERIES

ARITHMETIC

Series: it is the sum of a sequence

Sigma Notation: $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n$

Arithmetic Series: it is the sum of terms of an arithmetic sequence.

$$S_n = \sum_{i=1}^{n} a_i = \frac{(a_1 + a_n)}{2} \cdot n$$

- Proof:

$$\begin{array}{rcccccccl}
S_n = & a_1 & + & a_2 & + & \ldots & + & a_n & \\
S_n = & a_n & + & a_{n-1} & + & \ldots & + & a_1 & \\
\hline
2S_n = & (a_1 + a_n) & + & (a_1 + a_n) & + & \ldots & + & (a_1 + a_n) & \Rightarrow
\end{array}$$

$$2S_n = (a_1 + a_n) \cdot n \quad \Rightarrow \quad S_n = \frac{n \cdot (a_1 + a_n)}{2}$$

SERIES

GEOMETRIC

Geometric Series: it is the sum of the terms of a geometric sequence.

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} a_1 \cdot r^{n-1} \\
&= a_1 r^0 + a_1 r + a_1 r^2 + \ldots + a_1 r^n \\
&= a_1 (1 + r + r^2 + \ldots + r^n) \\
&= a_1 \cdot \frac{1 - r^n}{1 - r}, \text{ for } r < |1|
\end{aligned}
$$

Geometric Series: it is the sum of the terms of a geometric sequence.

- Proof: $1 + r + r^2 + ... + r^n = \frac{1}{1-r}$

Let:

$$\begin{array}{cccccccccccl} 1 & + & r & + & r^2 & + & ... & + & r^n & & = S_n & \text{Multiply by r} \\ & & r & + & ... & + & ... & + & r^n & +r^{n+1} & = rS_n & \text{substract} \\ \hline 1 & & & & & & & & & -r^{n-1} & = (1-r)S_n & \Rightarrow \end{array}$$

$$1 - r^{n-1} = (1-r)S_n \Rightarrow S_n = \frac{1-r^n}{1-r}$$

Infinite Geometric Series:(formula)

https://upload.wikimedia.org/wikipedia/commons/2/27/Logarithm$_G IF.g$

THE ZERO

- **Definition**
  - Number: 'zero' is a number representing no amount
  - Placeholder: it also is a digit indicating there is no quantity. e.g.: $502 \neq 52$ here means there are no tens.
- **Characteristics:**
  - It is not negative nor positive
  - It is an even number. $0/2 = 0$ so there is no remainder
  - It is also an idea: if there is nothing to count how can we count it?

THE ZERO

PROPERTIES

Property	Example
$a \pm 0 = a$	$4 \pm 0 = 4$
$0 \times / : a = 0$	$0 \times / : 4 = 0$
$a / 0 = undefined$	$4 \times / : 0 = undefined$
$0^a = 0 \; (a > 0)$	$0^4 = 0$
$0^0 = indeterminate$	$0^0 = indeterminate$
$0^a = undefined \; (a < 0)$	$0^{-4} = undefined$
$0! = 1$	$0! = 1$

https://www.mathsisfun.com/numbers/dividing-by-zero.html

INFINITY

- **Definition:** Infinity is something without an end.
- **Clarifications:**
  - Infinity does not grow, it does **not do** anything, it just **is**
  - It is not a **real number**
  - Infinity is just an **idea**
- **Using infinity:** Sometimes we can use infinity as if it were a real number, but it is not a real number. e.g.: $1 + \infty = \infty$

INFINITY

Properties
$\infty + \infty = \infty$
$\infty \times \infty = \infty$
$x + \infty = \infty$
$x \times \infty = -\infty \ (x < 0)$

Undefined Operation
$0 \times \infty$
$\infty - \infty$
∞ / ∞
∞^0
1^∞