Calculus

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LIMITS
INTUITION

Limit Intuition: We can get $f(x)$ as close to L 'as we want' by getting x sufficiently close to a.

Sometimes it is not possible to work out what the value of a function is, it might be indeterminate. So instead we work out the value as we get closer and closer but without actually being 'there'.

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INTUITION

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Sometimes it is not possible to work out what the value of a function is, it might be indeterminate. So instead we work out the value as we get closer and closer but without actually being 'there'.

$$\frac{x^2-1}{x-1} = \text{ undefined for } x = 1 \Rightarrow \text{but the limit} \lim_{x\to 1} \frac{x^2-1}{x-1} = 2 \text{ is defined}$$

LIMITS

- **Approach from the left/right:** functions need checking the limit from both sides to make sure it actually exists
  - Approach from the left: $\lim_{x \to a^-} f(x)$
  - Approach from the right: $\lim_{x \to a^+} f(x)$
- **Existence:** A limit L exists if the limit from the left is the same that the one from the right.

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x) \text{ for } a \neq \pm\infty$$

If the function is defined only over an interval, for extrema points it is only needed to check one of the sides.

Properties of limits: or limits of combined functions. Now define:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M$$

Then the properties are:

$$\begin{array}{ccccc}
\lim_{x\to c} f(x)+g(x) & = & \lim_{x\to c} f(x)+\lim_{x\to c} g(x) & = & L+M \\
\lim_{x\to c} f(x)-g(x) & = & \lim_{x\to c} f(x)-\lim_{x\to c} g(x) & = & L-M \\
\lim_{x\to c} f(x)\cdot g(x) & = & \lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x) & = & L\cdot M \\
\lim_{x\to c} f(x)/g(x) & = & \lim_{x\to c} f(x)/\lim_{x\to c} g(x) & = & L/M \\
\lim_{x\to c} kf(x) & = & k\lim_{x\to c} f(x) & = & k\cdot L
\end{array}$$

LIMITS

Unbounded limits (vertical asymptotes): it is encountered when the function $f(x)$ approaches ∞ as x tends to a point:

$$\lim_{x \to c} f(x) = \pm\infty$$

But don't be fooled by the "=". We cannot actually get to infinity, but in "limit" language the limit is infinity (which is really saying the function is limitless).

LIMITS

Limits at infinity (Horizontal asymptotes): it is the limit of a function as x approaches infinity.
It is not possible to say what $\frac{1}{\infty}$ is, but it is possible to work out what happens when x gets larger, $\lim_{x\to\infty} 1/x = 0$

- Rational: https://www.khanacademy.org/math/calculus-home/limits-and-continuity-calc/limits-at-infinity-calc/v/more-limits-at-infinity
- Radical: https://www.khanacademy.org/math/calculus-home/limits-and-continuity-calc/limits-at-infinity-calc/v/limits-with-two-horizontal-asymptotes
- Trigonometric: https://www.khanacademy.org/math/calculus-home/limits-and-continuity-calc/limits-at-infinity-calc/v/limit-at-infinity-involving-trig-defined
- Difference: https://www.khanacademy.org/math/calculus-home/limits-and-continuity-calc/limits-at-infinity-calc/v/limits-infinity-algebra

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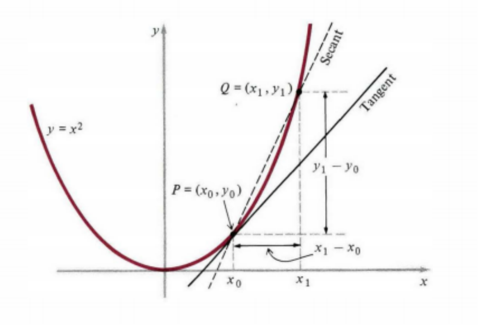
How to calculate the slope of the tangent at $P = (x_0, y_0)$

1. Choose a point $P = (x_0, y_0)$
2. Select a nearby point $Q = (x_1, y_1)$
3. Calculate the slope of the secant line m_{sec}

$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0}$$

4. Take the limit as $Q \to P$



Example: $y = x^2$

- Choose $P = (x_0, y_0)$
- Select $Q = (x_1, y_1)$
- Calculate m_{sec}

$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{x_1^2 - x_0^2}{x_1 - x_0}$$

- Take the limit

$$m = \lim_{P \to Q} m_{sec} = \lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0}$$

WARNING!!: at $x_1 = x_0$ the slope is not defined: $m_{sec} = \frac{0}{0}$, that's why we take the limit.

We must think of x_1 as coming very close to x_0 but *remaining distinct from it*

Solving the limit:

$$\lim_{x_1 \to x_0} \frac{y_1 - y_0}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{x_1^2 - x_0^2}{x_1 - x_0} =$$

$$= \lim_{x_1 \to x_0} \frac{(x_1 + x_0)(x_1 - x_0)}{x_1 - x_0} =$$

$$= \lim_{x_1 \to x_0} x_1 + x_0 = 2x_0$$

DERIVATIVES

DELTA NOTATION

$\Delta x = x_1 - x_0$: is the change in x going form the first value to the second or alternatively: $x_1 = x_0 + \Delta x$ adding a small amount to the first value.

Re writing m_{sec}

$$m_{sec} = \frac{x_1^2 - x_0^2}{x_1 - x_0} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}$$

$x_1 \to x_0$ is equivalent to $\Delta x \to 0$

solving the numerator:

$$\begin{aligned}(x_0 + \Delta x)^2 - x_0 =& x_0^2 + 2x_0\Delta x + (\Delta x)^2 - x_0^2 \\ =& 2x_0\Delta x + (\Delta x)^2 \\ =& \Delta x(2x_0 + \Delta x)\end{aligned}$$

And m_{sec} becomes: $m_{sec} = 2x_0 + \Delta x$, taking the limit:

$$m = \lim_{\Delta x \to 0} 2x_0 + \Delta x = 2x_0$$

Definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Procedure to compute derivatives:

1. write down the difference $f(x + \Delta x) - f(x)$ and simplify it to the point where Δx is a factor
2. Divide by Δx to form the *difference quotient*: $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
3. Evaluate the limit of the difference quotient as $\Delta x \to 0$

Example: $y = x^3$

STEP 1:

$$\begin{aligned} f(x+\Delta x) - f(x) &= (x+\Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3 \\ &= 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ &= \Delta x(3x^2 + 3x\Delta x + (\Delta x)^2) \end{aligned}$$

STEP 2:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2$$

STEP 3:

$$f'(x) = \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + (\Delta x)^2 = 3x^2$$

DERIVATIVES

NOTATION

All of these symbols are equivalent:

$$y' \qquad \frac{dy}{dx} \qquad f'(x) \qquad \frac{df(x)}{dx} \qquad \frac{d}{dx}f(x) \qquad D_x(f(x))$$

Why the fractions?

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

To indicate at a point:

$$\left.\frac{dy}{dx}\right|_{x=x_0}$$

DERIVATIVES

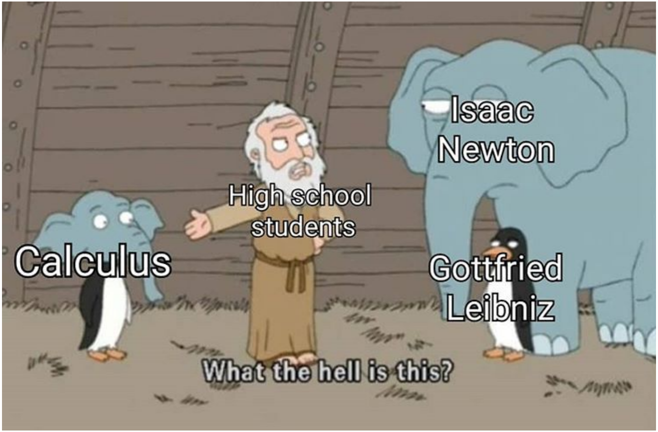
NOTATION

Why different notation? well...

DERIVATIVES

NOTATION

Why different notation? well...



CONSTANT: $y = c$

$$\frac{d}{dx}c = 0$$

Proof:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c-c}{\Delta x} = 0$$

POWER RULE: $y = x^n$ for $n \in \mathbb{Z}, n \neq 0$

$$\frac{d}{dx} x^n = n x^{n-1}$$

Proof:

$$\frac{dy}{dx} = \lim_{(\Delta x)\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \quad \ldots \text{expend } (x+\Delta x)^n$$

$$= \lim_{\Delta x \to 0} \frac{\left(x^n + n x^{n-1}\Delta x + \cdots n x(\Delta x)^{n-1} + (\Delta x)^n\right) - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{n x^{n-1}\Delta x + \frac{n(n-1)}{2!} x^{n-2}(\Delta x)^2 + \cdots n x \Delta x^{n-1} + (\Delta x)^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(n x^{n-1} + \frac{n(n-1)}{2!} x^{n-2}\Delta x + \cdots n x h^{n-2} + (\Delta x)^{n-1} \right)$$

$$= n x^{n-1}$$

CONSTANT TIMES A FUNCTION: $y = cf(x)$

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x)$$

Proof:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{cf(x+\Delta x) - cf(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0}\frac{c\left(f(x+\Delta x) - f(x)\right)}{\Delta x} \\ &= c\lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= cf'(x)\end{aligned}$$

SUM OF FUNCTIONS: $y = f(x) + g(x)$

$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof:

$$\begin{aligned}
\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0}\frac{\big(f(x+\Delta x) + g(x+\Delta x)\big) - \big(f(x) - g(x)\big)}{\Delta x} \\
&= \lim_{\Delta x \to 0}\frac{\big(f(x+\Delta x) - f(x)\big) + \big(g(x+\Delta x) - g(x)\big)}{\Delta x} \\
&= \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0}\frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

DERIVATIVES

COMPUTATION

PRODUCT RULE: $y = f(x) \cdot g(X)$

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = \frac{d}{dx} f(x) \cdot g(x) + f(x) \frac{d}{dx} g(x) = f'(x) g(x) + f(x) g'(x)$$

Proof:

$$\frac{d}{dx}\left[f(x) \cdot g(x)\right] =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) g(x+\Delta x) - f(x+\Delta x) g(x) + f(x+\Delta x) g(x) - f(x) g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\left[g(x+\Delta x) - g(x)\right] + \left[f(x+\Delta x) - f(x)\right] g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot f(x) =$$

$$= \underbrace{\lim_{\Delta x \to 0} f(x+\Delta x) \cdot}_{f(x)} \underbrace{\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}_{g'(x)} + \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}_{f'(x)} \cdot \underbrace{\lim_{\Delta x \to 0} g(x)}_{g(x)} =$$

$$= f'(x) g(x) + f(x) g'(x)$$

CHAIN RULE: $y = f\left(g(x)\right)$

$$\frac{d}{dx} f\left(g(x)\right) = \frac{df(x)}{dg(x)} \cdot \frac{dg(x)}{dx} = f'(g(x)) \cdot g'(x)$$

Proof:

Notice that for a continuous function $g(x)$ at a point:

$$as\ \Delta x \to 0 \Rightarrow \Delta g(x) \to 0$$

Then the result follows:

$$\frac{\partial f(g(x))}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \lim_{\Delta g \to 0} \frac{\Delta f}{\Delta g} \cdot \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial x}$$

QUOTIENT RULE: $y = \frac{f(x)}{g(X)}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x)\cdot g(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2} = \frac{f'(x)g(x) + f(x)g'(x)}{g(x)^2}$$

Proof:

$$\text{Notice that } \frac{f(x)}{g(x)} = f(x)\cdot g(x)^{-1}$$

Apply the product rule, for the second term use the power rule for $g(x)^{-1}$ then apply the chain rule.

Up to now all the functions have been of the form $y = f(x)$

However, it is not always obvious which is the independent variable: $F(x, y) = 0$

In these cases it is not straight forward what variable depends on which, but we can just assume that it does and differentiate implicitly.

Example:

$x^2 + y^2 = 25$	Using implicit differentiation w.r.t. y
$2x \cdot x' + 2y = 0$	Solving for x'
$x' = -\frac{y}{x}$	

Also we can use Implicit differentiation to prove:

$$\frac{\partial}{\partial x} x^n = n x^{n-1} \text{ for } n \in \mathbb{Q}$$

First we have y as a function of x: $\boldsymbol{y = x^n}$ where n is a rational number in the form $\boldsymbol{n = \frac{p}{q}}$. so we can write the equation as:

$$y = x^{\frac{p}{q}} \Leftrightarrow y^q = x^p$$

Assuming y depends on x and using implicit differentiation on the second term:

$$qy^{q-1}\frac{\partial y}{\partial x} = px^{p-1} \Leftrightarrow \frac{\partial y}{\partial x} = \frac{px^{p-1}}{qy^{q-1}} \qquad \text{Solving for } \frac{\partial y}{\partial x}$$

$$\Leftrightarrow \frac{\partial y}{\partial x} = \frac{px^{p-1}}{q\left(x^{\frac{p}{q}}\right)^{q-1}} \qquad \text{Substituting } y$$

$$\Leftrightarrow \frac{\partial y}{\partial x} = \frac{px^{p-1}}{qx^{p-\frac{p}{q}}} \qquad \text{Multiplying exponents}$$

$$\Leftrightarrow \frac{\partial y}{\partial x} = \frac{p}{q}x^{p-1-p+\frac{p}{q}}$$

$$\Leftrightarrow \frac{\partial y}{\partial x} = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}$$

EXPONENTIAL: $y = a^x$

$$\frac{dy}{dx} = a^x \ln a$$

Proof:
Using the definition of derivative:

$$\frac{da^x}{dx} = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \underbrace{\lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}}_{M(a)} = a^x M(a)$$

Now let's assume that $\exists! a = e | M(e) = 1$, Then:

$$\frac{d}{dx} e^x = e^x M(e) = e^x$$

LOGARITHM: $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

Proof:

Remember that $y = \ln x \iff e^y = x$, so:

$$e^y = x \qquad \text{Using implicit differentiation}$$

$$\frac{d}{dx}e^y \cdot \frac{dy}{dx} = 1 \iff e^y \cdot \frac{dy}{dx} = 1 \qquad \text{re writing and Solving for y'}$$

$$\frac{dy}{dx} = \frac{1}{e^y} \iff \frac{dy}{dx} = \frac{1}{e^{lnx}} \qquad \text{Substituting for its value}$$

$$\frac{dy}{dx} = \frac{1}{x}$$

EXPONETIALS: WHACHT OUT!!! we derived $\frac{d}{dx}e^x$ but what about the more general form $\frac{d}{dx}a^x$?

Proof (continuation):
Rewrite a as e^{lna} then:

$$a^x = e^{lna^x} = e^{xlna}$$

$$\frac{d}{dx}a^x = lna\, e^{xlna} \qquad \text{Using implicit differentiation}$$

$$\frac{d}{dx}a^x = lna\left(e^{lna}\right)^x = a^x lna \qquad \text{Undoing the change}$$

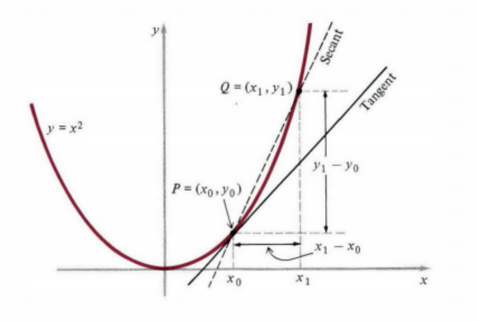
And notice that then $M(a) = lna$

The proof for the $log_a x$ in any base a is identical to the $\ln x$

INCREASE: What means for a function to be increasing?

$$\text{if } a < b \Rightarrow f(a) < f(b)$$

$$\text{if } f'(x) > 0 \Rightarrow f(x) \text{ is increasing}$$



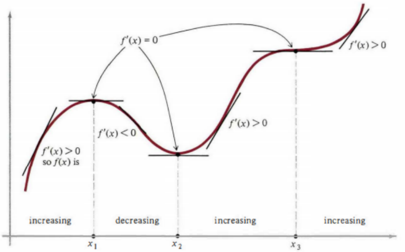
DECREASE:

$$\text{if } a < b \Rightarrow f(a) > f(b)$$

$$\text{if } f'(x) < 0 \Rightarrow f(x) \text{ is decreasing}$$

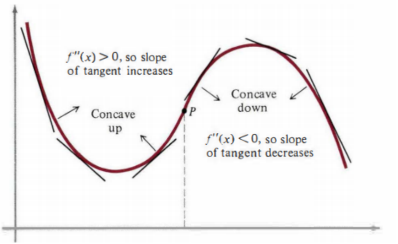
MAXIMUM/MINIMUM: Where does the function attains its local maxima and minima?

$$\text{if } f'(x_0) = 0 \Rightarrow f(x_0) \text{ is a critical point}$$



WHACHT OUT!!! $f'(x) = 0$ does not automatically mean that we are in a maximum or a minimum. I could be an inflection point

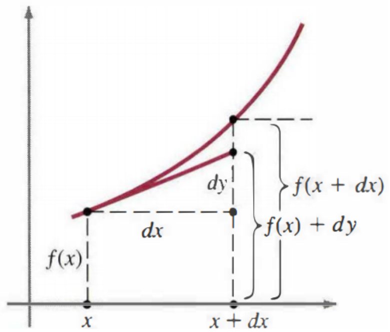
CONCAVITY AND POINTS OF INFLECTION: In what direction does the curve of the function bends?



- If $f''(x) > 0 \Rightarrow f(x)$ is Concave-up and attains a minimum
- If $f''(x) = 0 \Rightarrow f(x)$ is neither and possibly an inflection point
- If $f''(x) < 0 \Rightarrow f(x)$ is Concave-down and attains a maximum

APPROXIMATIONS:

$$f(x+dx) \approx f(x) + f'(x)\{(x+dx)-x\}\ \text{, for } x \approx x+dx$$



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INTEGRALS

INTUITION

ANTIDERIVATIVE: it is another name for integral.

They can be thought of the reverse operation of the derivative of $F(x)$

$$F'(x) = f(x) \iff F(x) + C = \int f(x)dx$$

By the very operation of the derivative, constants disappear. At the time of integration we have to take them back.

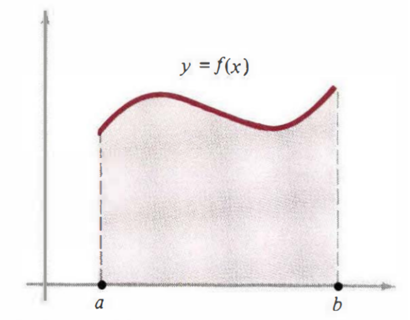
Example:

$$f(x) = x^3 \iff F(x) = \frac{x^4}{4} + C$$

INTEGRALS

INTUITION

AREA: **Definite** Integrals can be thought of as the area under the curve



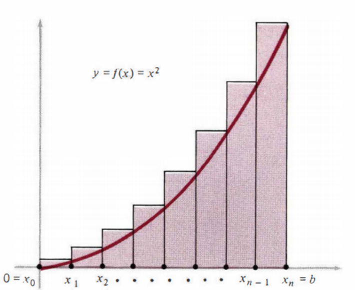
WHACHT OUT!!! Indefinite and definite integrals are two completelly different objects, they must not be confused.

INTEGRALS

RIEMAN SUMS

It is difficult to measure the area under a curve, but we can approximate it using rectangles

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



Of course, there is going to be some error, that can be avoided doing the intervals *"as small as possible"*

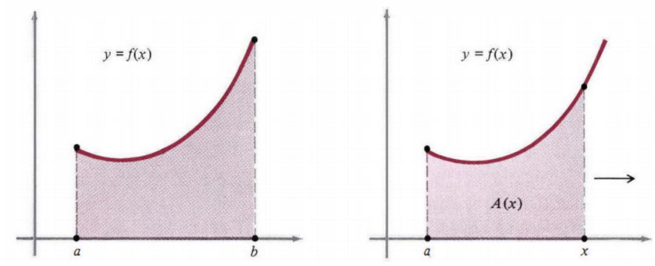
$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{\Delta x}{n}$$

FUNDAMENTAL THEOREM OF CALCULUS II: Let $f(x)$ be a continuous non-negative function in a close interval $[a, b]$. Then:

$$F(x) = \int_a^x f(t)\,dt \text{ or } \quad F'(x) = f(x)$$

PROOF:

$$\Delta F = F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(t)dt \approx f(x)\Delta x$$



PROOF:

$$\Delta F = F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(t)\,dt \approx f(x)\Delta x$$

Then:

$$\Delta F(x) \approx f(x)\Delta x \Longleftrightarrow \frac{\Delta F(x)}{\Delta x} \approx f(x)$$

Taking the limit as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta F(x)}{\Delta x} = f(x) \Longleftrightarrow F'(x) = f(x)$$

FUNDAMENTAL THEOREM OF CALCULUS I: Let $f(x)$ be a continuous non-negative function in a close interval $[a,b]$. Then:

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

PROOF: Since integration give us not only a function but a family of them, we can define:

$$G(x) = \int_a^x f(t)\,dt \overset{byFTCII}{\Longrightarrow} G'(x) = f(x)$$

since $G'(x) = f(x) = F'(x)$, we have $(G(x) - F(x))' = 0$

PROOF:

$$\text{then } G(x) - F(x) = C$$

To evaluate C, we evaluate at $x = a$, since $G(a) = 0$:

$$C = -F(a)$$

Then evaluate the function $G(x)$ at $x = b$ and use the value of C above:

$$G(b) = F(b) - F(a) \iff \int_a^b f(t)\,dt = F(b) - F(a)$$

INDEFINITE INTEGRALS:

$$\int c\, f(x)\, dx = c \int f(x)\, dx$$

$$\int \left[f(x) + g(x) \right]\, dx = \int f(x) dx + \int g(x)\, dx$$

INTEGRALS

PROPERTIES

DEFINITE INTEGRALS:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b \left[f(x)+g(x)\right]\,dx = \int_a^b f(x)dx + \int g(x)\,dx$$

DEFINITE INTEGRALS:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ and } \frac{d}{dx}\int_x^b f(t)dt = -f(x)$$

$$\text{if } f(x) \geq g(x), \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$\text{if } f(x) \leq 0, \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \leq 0$$

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

ANTIDERIVATIVE: Some integrals are easy to work out because they are just the opposite operation of the derivative.

$$\int_a^b e^x \, dx = e^x\big]_a^b + c \qquad \int_a^b \frac{1}{x} \, dx = \,ln x\big]_a^b + c$$

$$\int_a^b \sin x \, dx = -\cos x\big]_a^b + c \qquad \int_a^b \cos x \, dx = \sin x\big]_a^b + c$$

$$\int_a^b x^n \, dx = \frac{x^{n+1}}{n+1}\Big]_a^b + c$$

SUBSTITUTION: Let $F(x)$ be a non-negative and differentiable function and $g(x)$ a differentiable function in a close interval $[a, b]$. Furthermore let $y = F(g(x))$, then by the chain rule:

$$y' = \frac{dF\left(g(x)\right)}{dx} = F'\left(g(x)\right)g'(x) = f\left(g(x)\right)g'(x)$$

Integrating:

$$y = \int_a^b y' dx = \int_a^b f\left(g(x)\right)g'(x)dx$$

Now let:

$$u = g(x) \text{ and}$$
$$du = g'(x)dx$$

Substituting these values into the integrand:

$$y = \int_a^b y'dx = \int_a^b f\Big(\underbrace{g(x)}_{=u}\Big)\underbrace{g'(x)dx}_{=du}$$
$$= \int_{g(a)}^{g(b)} f(u)du$$
$$= F(u)]_{g(a)}^{g(b)} = F(g(x))\Big]_a^b + C$$

Example:

$$f(x) = \frac{\ln x}{x}$$

$$F(x) = \int_1^2 \frac{\ln x}{x} dx = \int_1^2 \ln x \cdot \frac{1}{x} dx$$

Now let:

$$u = \ln x \text{ and } du = \frac{1}{x} dx$$

$$u(1) = \ln 1 = 0 \text{ and } u(2) = \ln 2$$

Substituting:

$$F(x) = \int_1^2 \ln x \frac{1}{x} dx = \int_{u(1)}^{u(2)} u\, du = \left.\frac{u^2}{2}\right]_0^{\ln 2} = \left.\frac{1}{2}(\ln x)^2\right]_0^2 + C$$

BY PARTS: Let $f(x)$ and $g(x)$ be two non-negative and differentiable functions close interval $[a, b]$. Furthermore let $y = f(x)g(x)$, then by the product rule:

$$y' = \frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

Integrating:

$$\int_a^b \frac{d}{dx} f(x)g(x)dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

INTEGRALS

COMPUTATION

By the FTC II:

$$f(x)g(x)\Big]_a^b = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

Solving for $\int f(x)g'(x)\,dx$:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

INTUITION: the main object is to make $f(x)$ into something simpler, whilst letting $g(x)$ to remain in something similar or not more complicated.

Example:

$$f(x) = x\ \cos x$$
$$F(x) = \int_0^{\frac{\pi}{2}} x\ \cos x\ dx$$

Now let:

$$f(x) = x \text{ and } g'(x) = \cos x\ dx \text{ then:}$$
$$f'(x) = 1 \text{ and } g(x) = \sin x$$

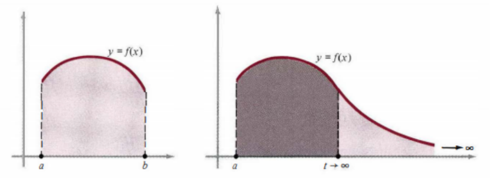
Integrating by parts:

$$\int_0^{\frac{\pi}{2}} x \cos x\ dx = x\ \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\ dx = x \sin x + \cos x]_0^{\frac{\pi}{2}} + C$$

IMPROPER INTEGRALS: are integrals of the form:

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

In which one (or both) of the limits of integration is infinite and the integrand $f(x)$ is assumed to be continuous on the unbounded interval $a \leq x < \infty$.



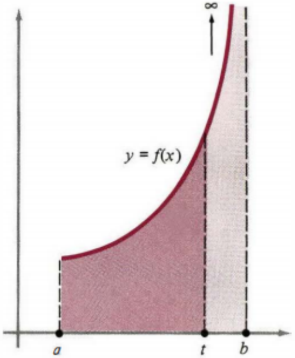
INTEGRALS

OTHER TYPES

IMPROPER INTEGRALS: are integrals of the form:

$$\int_a^b f(x)dx = \lim_{b\to t}\int_a^t f(x)dx$$

In which $f(x)$ becomes infinite as x approaches b



IMPROPER INTEGRALS: can be:

- **Convergent**: if the improper integral tends to a finite number
- **Divergent**: if the improper integral tends to infinity

Examples: convergent integrals

$$\int_0^{\infty} e^{-x} dx = -\left[e^{-x}\right]_0^{\infty} = -\lim_{b\to\infty}\left[e^{-x}\right]_0^{b} = -0+1 = 1+C$$

$$\int_0^{1} x^{-\frac{1}{2}} dx = 2\left[x^{\frac{1}{2}}\right]_0^{1} = 2\left[1-0\right] = 2+C$$

Examples: divergent integrals

$$\int_0^{\infty} \frac{1}{x} dx = \ln x]_1^{\infty} = \ln \infty - \ln 1 = \infty - 0 = \infty$$

$$\int_0^1 x^{-2} dx = -\left[\frac{1}{x}\right]_0^1 = -1 + \lim_{x \to 0^+} \frac{1}{x} = -1 + \infty = \infty$$

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POWER SERIES

POWER SERIES: they are series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where the coefficients of a_n are constants and x is a variable. Notice that power series are themselves functions $(f(x))$

Example:

$$\sum x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \text{ for} x < |1|$$

POWER SERIES

As well as polynomials, that are finite, power series share some interesting characteristics. It can be said that within the radius of convergence:

- Power series are continuous
- Are differentiable
- Are integrable

POWER SERIES

TAYLOR'S RULE

TAYLOR POWER SERIES: we have seen that power series are functions in their own right, some of them with a close form solution, such as: $\sum x^n = \frac{1}{1-x}$.

We would like to know if when we encounter a function, it can be expressed in terms of a power series. It turns out that it is possible to do so within the radius of convergence.

Assume we have any $f(x)$ and we would like to write in the form of a power series, i.e.:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

POWER SERIES

TAYLOR'S RULE

Assume we have any $f(x)$ and we would like to write it in the form of a power series, i.e.:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

As seen in previous slide, infinitely many derivatives can be taken:

$$\begin{aligned} f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \dots \\ f''(x) &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 \dots \\ &\dots \\ f^n(x) &= n! a_n + \text{Terms containing } x \text{ as a factor} \end{aligned}$$

Now notice that at $x = 0$, the terms that share x as a factor cancel, having:

$$
\begin{aligned}
f(0) &= a_0 && \Rightarrow a_0 = f(0) \\
f'(0) &= a_1 && \Rightarrow a_1 = f'(0) \\
f''(0) &= 2a_2 && \Rightarrow a_2 = \frac{1}{2} f''(0) \\
&\dots \\
f^n(0) &= n! a_n && \Rightarrow a_n = \frac{1}{n!} f^n(0)
\end{aligned}
$$

POWER SERIES

TAYLOR'S RULE

Substituting back into the original equation:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^3(0)}{3!}x^3 + ... + \frac{f^n(0)}{n!}x^n + ...$$
$$= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

POWER SERIES

TAYLOR'S RULE

Example: take $\ln(1+x)$
We would like to write it in the form: $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, then:

$$f(0) = \ln 1 = 0 \qquad \Rightarrow a_0 = 0$$

$$f'(0) = \left.\frac{1}{1+x}\right|_{x=0} = 1 \qquad \Rightarrow a_1 = 1$$

$$f''(0) = \left.\frac{-1}{(1+x)^2}\right|_{x=0} = -1 \qquad \Rightarrow a_2 = -\frac{1}{2}$$

$$\ldots$$

$$f^n(0) = (-1)^{n-1}\left.\frac{(n-1)!}{(1+x)^n}\right|_{x=0} = (-1)^{n-1}(n-1)! \qquad \Rightarrow a_n = (-1)^{n-1}\frac{1}{n}$$

POWER SERIES

TAYLOR'S RULE

Example: $\ln(1+x)$
Substituting back into Taylor's formula:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^n \frac{x^{n+1}}{n+1}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Look at the gif for $\ln(1+x)$:
https://upload.wikimedia.org/wikipedia/commons/2/27/Logarithm_GIF.gif

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MULTIVARIATE CALCULUS

INTRODUCTION

Many functions do not depend only on one variable but in an undefined number of them, e.g.:

$$z = f(x, y)$$

Is a function that depends only on x and y. Of course a function might have any number of variables:

$$z = f(\boldsymbol{x}) = f(x_1, x_2, ..., x_n)$$

This specific arrange of variables is called a **vector**. As such, we can define bold **x** as this vector, hence:

$$\boldsymbol{x} = (x_1, x_2, ..., x_n)$$

DOMAIN: the domain is all the points $P = (x_{1_0}, x_{2_0}, ..., x_{n_0})$ in the plane for which the function $z = f(\mathbf{x})$ is defined
Example 1:

$$z = f(x, y) = \frac{1}{x - y}$$

This function is not define for all values where $x = y$
Example 2:

$$w = g(\boldsymbol{x}) = \sqrt{9 - x^2 - y^2}$$

This function is not define for all values where $x^2 + y^2 \geq 9$

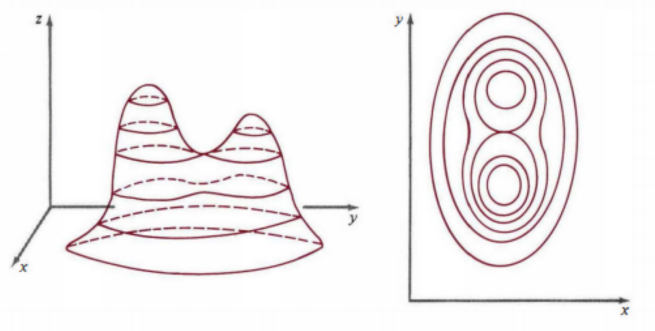
MULTIVARIATE CALCULUS

LEVEL CURVES

LEVEL CURVE: is the reflected line over the xy-plane where the function takes the same value:

$$z = f(x, y) = c$$

The collection of level curves is called the **contour-map**



PARTIAL DERIVATIVE: is the derivative of a multivariate function w.r.t. one of its variables. The key idea is to allow one variable change while keeping the rest constant:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = f_x(x,y)$$
$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = f_y(x,y)$$

And in general:

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_i+\Delta x_i, \boldsymbol{x}_{-i}) - f(\boldsymbol{x})}{\Delta x_i} = f_{x_i}(\boldsymbol{x})$$

Example:

$$f(x,y) = x^4 + 3x^2y^3 - \ln(2x^2y)$$
$$f_x = 4x^3 + 6xy^3 - \frac{2}{x}$$
$$f_y = 9x^2y^2 - \frac{1}{y}$$

NOTATION: $\frac{\partial z}{\partial x}$ this limit (if it exist) is the *partial derivative of z w.r.t. x*. The most common notations are:

$$\frac{\partial z}{\partial x}, \quad z_x, \quad \frac{\partial f}{\partial x}, \quad f_x, \quad f_x(x,y)$$

As with functions of one variable, multivariate functions are functions on their own right and we can expect to have *second order partial derivatives* w.r.t. x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

More interestingly $f_{xy} = f_{yx}$

Example:

$$f_x = 4x^3 + 6xy^3 - \frac{2}{x} \qquad f_{yx} = 18xy^2$$

$$f_y = 9x^2y^2 - \frac{1}{y} \qquad f_{xy} = 18xy^2$$

TANGENT PLANE: The concept of tangent plane to a surface corresponds to the concept of tangent line to a curve. So the tangent plane of a surface at a point is the plane that "best approximates" the surface at that point.

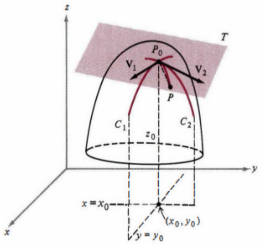


Figure: Tangent plane

Tangent line	**Tangent plane**
$m(x-x_0)+(y-y_0)=0$	$a(x-x_0)+b(y-y_0)+(z-z_0)=0$
$f'(x_0)(x-x_0)+\big(f(x)-f(x_0)\big)=0$	$f_x(x-x_0)+f_y(y-y_0)+\big(f(x,y)-f(x_0,y_0)\big)=0$

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IMPLICIT FUNCTION THEOREM

CHAIN RULE

Let $w = f(x, y)$ be a differentiable function in a closed interval. Let also $x = g(t)$ and $y = h(t)$ be continuous functions in the same interval. Then:

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

In general for $w = f(\boldsymbol{x})$:

$$\frac{\partial f(\boldsymbol{x})}{\partial t} = \frac{\partial f(\boldsymbol{x})}{\partial x_1}\frac{\partial x_1}{\partial t} + \ldots + \frac{\partial f(\boldsymbol{x})}{\partial x_n}\frac{\partial x_n}{\partial t}$$

IMPLICIT FUNCTION THEOREM

THEOREM

THEOREM: Let $F(x, y)$ have continuous partial derivatives throughout some neighbourhood of a point (x_0, y_0), and assume that $F(x_0, y_0) = c$ and $F_y(x_0, y_0) \neq 0$. Then there is an interval I about x_0 with the property that there exists exactly one differentiable function $y = f(x)$ defined on I such that $y_0 = f(x_0)$ and:

$$F\left[x, f(x)\right] = c$$

Further, the derivative of this function is given by the formula

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

and is therefore continuous.

Example: consider $F(x, y) = x^2 y^5 - 2xy + 1 = 0$ Taking the partial derivatives:

$$F_x(x, y) = 2xy^5 - 2y$$
$$F_y(x, y) = 5x^2 y^4 - 2x$$

Then:

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2xy^5 - 2y}{5x^2 y^4 - 2x}$$

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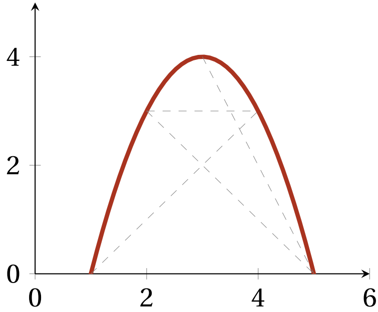
CONVEX AND CONCAVE FUNCTIONS

INTUITION

CONCAVE FUNCTION: is a function where no line segment joining two points on the graph lies **above** the graph at any point.

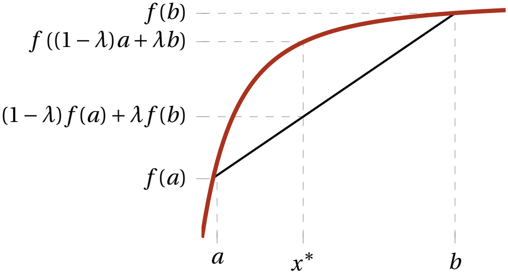


CONVEX AND CONCAVE FUNCTIONS

DEFINITION

DEFINITION: Let $f(x)$ be a function defined on the interval I. Then $f(x)$ is said to be **concave** if $\forall a, b \in I$, and $\forall \lambda \in [0,1]$ we have:

$$f((1-\lambda)a + \lambda b) \geq (1-\lambda) f(a) + \lambda f(b)$$

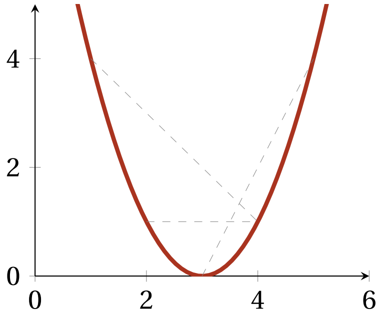


where $x* = (1-\lambda)a + \lambda b$

CONVEX AND CONCAVE FUNCTIONS

INTUITION

CONVEX FUNCTION: is a function where no line segment joining two points on the graph lies **below** the graph at any point.



DEFINITION: Let $f(x)$ be a function defined on the interval I. Then $f(x)$ is said to be **convex** if $\forall a, b \in I$, and $\forall \lambda \in [0,1]$ we have:

$$f((1-\lambda)a+\lambda b) \leq (1-\lambda) f(a) + \lambda f(b)$$

CONVEX AND CONCAVE FUNCTIONS

JENSEN'S INEQUALITY

A function $f(x)$ of a single variable defined on the interval I is **concave** if and only if $\forall n \geq 2$:

$$f(\lambda_1 x_1 + ... + \lambda_n x_n) \geq \lambda_1 f(x_1) + ... + \lambda_n f(x_n)$$

$$\forall x_1, ..., x_n \in I \text{ and } \forall \lambda_1, ..., \lambda_n \geq 0 \mid \sum_{i=1}^{n} \lambda_i = 1$$

A function $f(x)$ of a single variable defined on the interval I is **convex** if and only if $\forall n \geq 2$:

$$f(\lambda_1 x_1 + ... + \lambda_n x_n) \leq \lambda_1 f(x_1) + ... + \lambda_n f(x_n)$$

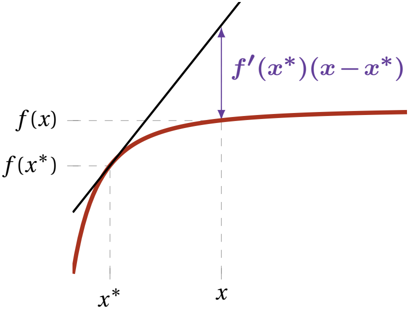
$$\forall x_1, ..., x_n \in I \text{ and } \forall \lambda_1, ..., \lambda_n \geq 0 \mid \sum_{i=1}^{n} \lambda_i = 1$$

CONVEX AND CONCAVE FUNCTIONS

DIFFERENTIABLE FUNCTIONS

DEFINITION: The differentiable function $f(x)$ of a single variable defined on an open interval I is **concave** on I if and only if:

$$f(x) - f(x^*) \leq f'(x^*)(x - x^*)$$



INTUITION: The graph of the function $f(x)$ lies below the the any tangent line

DEFINITION: The differentiable function $f(x)$ of a single variable defined on an open interval I is **convex** on I if and only if:

$$f(x) - f(x^*) \geq f'(x^*)(x - x^*)$$

INTUITION: The graph of the function $f(x)$ lies below the the any tangent line

CONVEX AND CONCAVE FUNCTIONS

TWICE-DIFFERENTIABLE FUNCTIONS

PROPOSITION: A twice-differentiable function $f(x)$ of a single variable defined on the interval I is:

- **Concave:** if and only if $f''(x) \leq 0$ for all x in the interior of I
- **Convex:** if and only if $f''(x) \geq 0$ for all x in the interior of I

INTUITION: For a concave (convex) function, the slope of the tangent line to a point becomes lesser as we move along the x-axis